# The Poisson Distribution

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## Probability Mass Function

The probability mass function of the Poisson distribution with mean $\mu$ is

$$\Pr\{Y = k \mid \mu\} = \frac{e^{-\mu}\mu^k}{k!} \qquad \text{for } k = 0, 1, 2, \ldots.$$

The Poisson distribution is discrete, like the binomial distribution, but has only a single parameter $\mu$ that is both the mean and the variance.

## The Poisson Distribution

The Poisson distribution arises in many biological contexts. Examples of random variables for which a Poisson distribution might be reasonable include:

- the number of bacterial colonies in a Petri dish;
- the number of trees in an area of land;
- the number of offspring an individual has;
- the number of nucleotide base substitutions in a gene over a period of time;

## Example

Suppose the number of individual plants of a given species we expect to find in a one meter square quadrat follows the Poisson distribution with mean $\mu = 10$. Find the probability of finding exactly 12 individuals.

$Y \sim \text{Poisson}(10)$.

$$\Pr\{Y = 12 \mid \mu = 10\} = \frac{e^{-10}10^{12}}{12!} \doteq 0.0948.$$

## Example in R

- In R, you can compute Poisson probabilities with the function `dpois`.
- For the previous example, try the following.

```
> dpois(12, 10)
[1] 0.09478033
```

## Poisson approximation to the Binomial

- The Poisson distribution is good approximation to the binomial distribution when $p$ is small.
- The approximation is better for large $n$.
- If $p$ is small, then the binomial probability of exactly $k$ successes is approximately the same as the Poisson probability of $k$ with $\mu = np$.

## Example

Suppose that $Y \sim \text{Binomial}(1000, 0.01)$. Find $\Pr\{Y = 8\}$. The exact calculation is:

$$\Pr\{Y = 8\} = \frac{1000!}{8!992!}(0.01)^8(0.99)^{992} \doteq 0.112824$$

Working with large factorials can be messy. The Poisson approximation uses $\mu = 1000 \times 0.01 = 10$ and is:

$$\Pr\{Y = 8\} \approx \frac{e^{-10}\mu^8}{8!} \doteq 0.112599$$

## Example using R

Here is the same example using R.

```
> dbinom(8, 1000, 0.01)
[1] 0.1128241
> dpois(8, 1000 * 0.01)
[1] 0.1125990
```

# The Poisson Process

- The Poisson Process arises naturally under assumptions that are often reasonable.
- For the following, think of points as being exact times or locations.
- The assumptions are:
  - The chance of two simultaneous points is negligible;
  - The expected value of the random number of points in a region is proportional to the size of the region.
  - The random number of points in non-overlapping regions are independent.
- Under these assumptions, the random variable that counts the number of points has a Poisson distribution.
- If the expected rate of points is $\lambda$ points per unit length (area), then the distribution of the number of points in an interval (region) of size $t$ is $\mu = \lambda t$.

# Example

Suppose that we assume that at a location, a particular species of plant is distributed according to a Poisson process with expected density 0.2 individuals per square meter. In a nine square meter quadrat, what is the probability of no individuals?
**Solution:** The number of individuals has a Poisson distribution with mean $\mu = 9 \times 0.2 = 1.8$. The probability of this is

$$\Pr\{Y = 0 \,|\, \mu = 1.8\} = \frac{e^{-1.8}(1.8)^0}{0!} \doteq 0.165299$$

In R, we can compute this as

```
> dpois(0, 1.8)
[1] 0.1652989
```

# Example (cont.)

Find the probability of three or more individuals.
**Solution:** Instead of summing the probabilities from 3 to infinity, we can use the complement rule.

$$\Pr\{Y \geq 3\} = 1 - \Pr\{Y \leq 2\} = 1 - \Pr\{Y = 0\} - \Pr\{Y = 1\} - \Pr\{Y = 2\}$$

In R, this is found by one of two ways.

```
> 1 - ppois(2, 1.8)
[1] 0.2693789
> 1 - sum(dpois(0:2, 1.8))
[1] 0.2693789
```