Sums of Independent Random Variables

Consider the sum of two independent discrete random variables X and Y whose values are restricted to the non-negative integers. Let $f_X(\cdot)$ denote the probability distribution of X and $f_Y(\cdot)$ denote the probability distribution of Y. The distribution of their sum $Z = X + Y$ is given by the **discrete convolution formula**.

Theorem *Discrete Convolution Formula.* The random variable $Z = X + Y$ has probability distribution $f_Z(\cdot)$ given by

$$f_Z(z) \;=\; f_{X+Y}(z) = P(\,Z \;=\; z\,) = \sum_{x=0}^{z} f_X(x)f_Y(z-x)$$

for $z = 0, 1, \ldots$.

Proof: For each z, the event $[Z = z]$ is the union of the disjoint events $[X = x \;\text{ and }\; Y = z - x]$ for $x = 0, 1, ..., z$. Consequently,

$$\begin{aligned} P(\,Z \;=\; z\,) \;=\; f_Z(z) \quad &= \textstyle\sum_{x=0}^{z} P(X = x \;\text{ and }\; Y = z - x) \\ &= \textstyle\sum_{x=0}^{z} f_X(x)f_Y(z-x) \end{aligned}$$

where the last step follows by independence.

Let X_1 and X_2 be independent binomial random variables having the same probability of success. Their sum is again binomial.

Corollary 1 *Sum of Binomial Random Variables.* Let X_1 and X_2 be independent binomial random variables where X_i has a Binomial(n_i, p) distribution for $i = 1, 2$. Then

$$X_1 + X_2 \quad \text{has a binomial distribution with } n_1 + n_2 \text{ trials and probability of success } p$$

Let $X_1, X_2, ...X_k$ be independent binomial random variables where X_i has a Binomial(n_i, p) distribution for $i = 1, 2, ..., k$. Then

$$X_1 \;+\; X_2 \;+\; \cdots X_k \qquad \text{has a Binomial}(n_1 + n_2 + \cdots + n_k, p) \quad \text{distribution.}$$

Proof: By the discrete convolution formula, $Z = X_1 + X_2$ has probability distribution

$$P(X_1 + X_2 = z) = f_Z(z) = \sum_{x=0}^{z} f_{X_1}(x) f_{X_2}(z - x)$$

so

$$f_Z(z) = \sum_{x=0}^{z} \binom{n_1}{x} p^x (1-p)^{n_1 - x} \binom{n_2}{z-x} p^{z-x} (1-p)^{n_2 - (z-x)}$$

$$= p^z (1-p)^{n_1 + n_2 - z} \sum_{x=0}^{z} \binom{n_1}{x} \binom{n_2}{z-x}$$

Now, equating the coefficients of s^y in the binomial expansion of both sides of

$(1+s)_1^n (1+s)_2^n = (1+s)^{n_1+n+2}$

we conclude that

$$\sum_{x=0}^{z} \binom{n_1}{x} \binom{n_2}{z-x} = \binom{n_1 + n_2}{z}$$

The case for several binomial random variables follows by induction.

Remark: Note that the sample sizes add but the success probability remains the same.

Corollary 2 *Sum of Poisson Random Variables.* Let X_1 and X_2 be independent Poisson random variables where X_i has a Poisson (λ_i) distribution for $i = 1, 2$. Then

$$X_1 + X_2 \quad \text{has a Poisson distribution with} \quad \lambda_1 + \lambda_2$$

Let $X_1, X_2, ...X_k$ be independent Poisson random variables where X_i has a Poisson (λ_i) distribution for $i = 1, 2, ..., k$. Then

$$X_1 + X_2 + \cdots X_k \quad \text{has a Poisson}(\lambda_1 + \lambda_2 + \cdots + \lambda_k) \quad \text{distribution.}$$

Proof: By the discrete convolution formula, $Z = X_1 + X_2$ has probability distribution

$$P(X_1 + X_2 = z) = f_Z(z) = \sum_{x=0}^{z} f_{X_1}(x) f_{X_2}(z - x)$$

so

$$f_Z(z) = \sum_{x=0}^{z} \frac{\lambda_1^x}{x!} e^{-\lambda_1} \frac{\lambda_2^{z-x}}{(z-x)!} e^{-\lambda_2}$$
$$= e^{-(\lambda_1 + \lambda_2)} \sum_{x=0}^{z} \frac{\lambda_1^x}{x!} \frac{\lambda_2^{z-x}}{(z-x)!}$$

Use the binomial formula

$$(a+b)^m = \sum_{x=0}^{m} \binom{m}{x} a^x b^{m-x}$$

with $m = z, a = \lambda_1$ and $b = \lambda_2$ after multiplying and dividing by $z!$, to conclude that

$$\sum_{x=0}^{z} \frac{\lambda_1^x}{x!} \frac{\lambda_2^{z-x}}{(z-x)!} = \frac{(\lambda_1 + \lambda_2)^z}{z!}$$

and the result is established.
Remark: Note that the rate parameters λ_i add.