Completely Randomized (One-Way) Design

Model

\[ Y_{ij} = \mu_i + \epsilon_{ij} \]

or

\[ Y_{ij} = \mu + \alpha_i + \epsilon_{ij} \]

where

\( i = 1, \ldots, k \) indexes treatment

\( j = 1, \ldots, n \) indexes experimental unit (plot) for each treatment

\( \epsilon_{ij} \sim N(0, \sigma^2_\epsilon) \) corresponds to error from plot to plot

ANOVA Table

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>SS</th>
<th>MS</th>
<th>E(MS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment</td>
<td>k-1</td>
<td>SSTrt</td>
<td>MSTrt</td>
<td>( \sigma^2 + n \sum_{i=1}^{k} \alpha^2_i / (k-1) )</td>
</tr>
<tr>
<td>Error</td>
<td>k(n-1)</td>
<td>SSErr</td>
<td>MSErr</td>
<td>( \sigma^2 )</td>
</tr>
<tr>
<td>Total</td>
<td>kn-1</td>
<td>SSTot</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where

\[ \text{SSTrt} = n \sum_{i=1}^{k} (\bar{y}_i - \bar{y}.)^2 = \frac{1}{n} \sum_{i=1}^{k} y^2_i - CF \]

\[ \text{SSErr} = \sum_{i=1}^{k} \sum_{j=1}^{n} (y_{ij} - \bar{y}_i)^2 \]

\[ \text{SSTot} = \sum_{i=1}^{k} \sum_{j=1}^{n} (y_{ij} - \bar{y}.)^2 = \sum_{i=1}^{k} \sum_{j=1}^{n} y^2_{ij} - CF \]

\[ CF = \frac{1}{kn} \bar{y}^2 \]

We use \( F = \text{MSTrt}/\text{MSErr} \) to test \( H_0 : \mu_1 = \cdots = \mu_k \) vs \( H_A : \) not \( [\mu_1 = \cdots = \mu_k] \). Equivalently, we can write \( H_0 \) as: \( H_0 : \alpha_i = 0 \forall i \).

Contrasts

- A contrast can be tested using

\[ T = \frac{\sum c_i \bar{y}_i}{\sqrt{\sum c_i^2/n}} = \frac{\sum c_i \bar{y}_i}{s \sqrt{\sum c_i^2/n}} \]

- The sum of squares for a contrast is given by

\[ \frac{(\sum c_i \bar{y}_i)^2}{\sum c_i^2/n} \]
An Example of One-Way ANOVA

Consider 3 groups with seven observations per group.

The raw data are:

<table>
<thead>
<tr>
<th></th>
<th>y1</th>
<th>y2</th>
<th>y3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y1</td>
<td>18.2</td>
<td>17.4</td>
<td>15.2</td>
</tr>
<tr>
<td>y2</td>
<td>20.1</td>
<td>18.7</td>
<td>18.8</td>
</tr>
<tr>
<td>y3</td>
<td>17.6</td>
<td>19.1</td>
<td>18.8</td>
</tr>
<tr>
<td>y4</td>
<td>16.8</td>
<td>16.4</td>
<td>16.5</td>
</tr>
<tr>
<td>y5</td>
<td>18.8</td>
<td>15.9</td>
<td>15.9</td>
</tr>
<tr>
<td>y6</td>
<td>19.7</td>
<td>17.7</td>
<td>17.1</td>
</tr>
<tr>
<td>y7</td>
<td>19.1</td>
<td>17.7</td>
<td>16.7</td>
</tr>
</tbody>
</table>

A stem and leaf display of these data looks like:

```
15.  9
16.  8  4  57
17.  6  47  71
18.  28  74  8
19.  71  1
20.  1
```

Summary statistics are:

<table>
<thead>
<tr>
<th></th>
<th>y1</th>
<th>y2</th>
<th>y3</th>
</tr>
</thead>
<tbody>
<tr>
<td>\bar{y}_1</td>
<td>130.3</td>
<td>123.6</td>
<td>117.9</td>
</tr>
<tr>
<td>\bar{y}_2</td>
<td>18.61</td>
<td>17.66</td>
<td>16.84</td>
</tr>
<tr>
<td>s^2_1</td>
<td>1.358</td>
<td>1.410</td>
<td>1.393</td>
</tr>
</tbody>
</table>

The sum of squares for these groups is:

\[
SS_{\text{Trt}} = \frac{\sum (\bar{y}_i - \bar{y})^2}{18} = 11.01
\]

The corresponding ANOVA table is:

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>SS</th>
<th>MS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trt</td>
<td>2</td>
<td>11.01</td>
<td>5.505</td>
</tr>
<tr>
<td>Error</td>
<td>18</td>
<td>24.96</td>
<td>1.387</td>
</tr>
<tr>
<td>Total</td>
<td>20</td>
<td>35.97</td>
<td></td>
</tr>
</tbody>
</table>

Note that the MS for Error is the same as the pooled estimate of variance, MSE = \(s^2_\epsilon = \frac{6s^2_1 + 6s^2_2 + 6s^2_3}{18} = 24.96/18 = 1.387\).

To test \(H_0: \mu_1 = \mu_2 = \mu_3\) we compute the test statistic:

\[ F = \frac{\text{MSTrt}}{\text{MSErr}} = \frac{5.505}{1.387} = 3.97 \]

and compare this with \(F_{2,18}\); the p-value is .037.

For the contrast \(\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 - \mu_3\) the standard error is

\[ s_\epsilon \sqrt{\frac{\sum c^2_i}{n}} = \sqrt{1.387} \sqrt{\frac{1^2 + 1^2 + (-1)^2}{7}} = .5452 \]

and so a t-test for \(H_0: \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 - \mu_3 = 0\) is given by:

\[ t = \frac{\frac{1}{2}\bar{y}_1 + \frac{1}{2}\bar{y}_2 - \bar{y}_3}{.5452} = \frac{1.2950}{.5452} = 2.38 \]

The sum of squares for this contrast is

\[ \frac{(\sum c_i \bar{y}_i)^2}{\sum c^2_i/n} = 7.80 \]

This leads to an alternative test for \(H_0: \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 - \mu_3 = 0\)

\[ F = \frac{\text{SSContrast}/1}{\text{SSErr}/18} = \frac{7.80}{1.387} = 5.62 \]

For the contrast \(\mu_1 - \mu_2 + 0\mu_3\) the sum of squares is 3.21.