Asymptotically (In)dependent Multivariate Maxima of Moving Maxima Processes

Janet E. Heffernan[†], Jonathan A. Tawn[†], and Zhengjun Zhang[‡]

[†]Department of Mathematics and Statistics, Lancaster University
Lancaster, LA1 4YF, UK
[‡]Department of Statistics, University of Wisconsin
Madison, WI 53706, USA

January 9, 2007

Abstract

Smith and Weissman introduced a M4 class of processes which are very flexible models for temporally dependent multivariate extreme value processes. However all variables in these M4 models are asymptotically dependent and what this paper does is to extend this M4 class in a number of ways to produce classes of models which are also asymptotically independent. We shall study properties of the proposed models. In particular, asymptotic dependence indexes, coefficients of tail dependence, and extremal indexes are derived for each case.

Keywords: Asymptotic (in)dependence, near independence, positive dependence, negative dependence, multivariate time series, extreme value theory.

2000 Mathematics Subject Classification: 60G70, 62H05

1 Introduction

1.1 General introduction

Examples of variables moving together are too many. For instance, the co-movements of prices of hundreds of stocks in any particular trading day in a financial market, and the co-movements of wind speeds, wave heights, water levels in a very broad sea area in any time period are typical examples of those variables. One main purpose of developing multivariate time series models is to explore the cross-sectional and serial dependence structures among the variables of interest.

In multivariate extreme value context, the main interests have been in constructing multivariate extreme value distributions. Theoretical probability treatments can be seen in Pickands (1981), Resnick (1987), etc. Joe (1997) is an excellent source of multivariate dependence models. Mari and Kotz (2001) deal with dependence concepts and measures. For serial dependence, Deheuvels (1983) defines the moving minimum (MM) process; Davis and Resnick (1989) study what they call the max-autoregressive moving average (MARMA) process of a stationary process; Hall, Peng, and Yao (2002) discuss moving maximum models. However, little efforts have been put on constructing models for cross-sectional and serial extreme dependencies.

Smith and Weissman (1996) extend Deheuvels' definition to the so called multivariate maxima of moving maxima (henceforth M4) process:

$$Y_{id} = \max_{l} \max_{k} a_{l,k,d} Z_{l,i-k}, \quad d = 1, \ldots, D, \ -\infty < i < \infty, \tag{1.1}$$

for nonnegative constants $\{a_{l,k,d}, l \geq 1, -\infty < k < \infty\}$ satisfying $\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} a_{l,k,d} = 1$ for $d = 1, \ldots, D$, and $\{Z_{lk}, l \geq 1, -\infty < k < \infty\}$ being an array of independent unit Fréchet random variables which have distribution form $\exp(-1/z), \ z > 0$.

In an autoregressive and moving average time series model, the random error is sometimes called the innovation variable or shock variable. It would be convenient to call Z_{li} as unit Fréchet max-shock variable, and hence we can have different kinds of max-shock variables.

The additional properties of this M4 class of processes and their financial applications and environmental applications are studied by Zhang (2002, 2003), Zhang and Smith (2004a, b), etc. This M4 class of processes is very flexible for data with asymptotic dependence, also regarded as extreme dependence or tail dependence. However all variables in these M4 models are asymptotically dependent. There are many examples among which data are not asymptotically dependent. They are nearly independent, positive dependent, asymptotically independent as witnessed by Ledford and Tawn (1996, 1997), Heffernan and Tawn (2004), Draisma, Drees, Ferreira, and de Haan (2004), Peng(1999), and many others. These new observed phenomena require new developments in the multivariate extreme value context. The motivation of this paper came from the need of building models for asymptotically independent variables, the further study of the flexible M4 model structure, and the need of computing two tail dependence measures (the asymptotic dependence index and the coefficient of tail dependence) between variables in practice. These two dependence measures and the extremal index are reviewed next.

1.2 The concept of asymptotic (in)dependence

Sibuya (1960) introduces asymptotic independence between two random variables with identical marginal distribution. De Haan and Resnick (1977) extend it to the case of multivariate random variables. The definition is given below.

Definition 1 *A bivariate random variable* $(X_1,\ X_2)$ *is called asymptotically independent if*

$$\lambda = \lim_{x \to x_F} \mathrm{P}(X_2 > x | X_1 > x) = 0, \tag{1.2}$$

where X_1 *and* X_2 *are identically distributed with* $x_F = \sup\{x \in \mathbb{R} :\ \mathrm{P}(X_1 \leq x) < 1\}$*;* λ *is also called the bivariate asymptotic dependence index which quantifies the amount of dependence of the bivariate upper tails. If* $\lambda > 0$*, then* $(X_1,\ X_2)$ *is called asymptotically dependent.*

If the joint distribution between X_1 and X_2 is known, we may be able to derive an explicit formula for λ. For example, when X and Y are normally distributed with correlation $\rho \in [-1, 1)$ then, $\lambda = 0$. When X_1 and X_2 have a standard bivariate t-distribution with ν degrees of freedom and correlation $\rho > -1$ then, $\lambda = 2\bar{t}_{\nu+1}(\sqrt{\nu+1}\sqrt{1-\rho}/\sqrt{1+\rho})$, where $\bar{t}_{\nu+1}$ is the tail of standard t distribution. Heffernan (2000), Embrechts, McNeil, and Straumann (2002) give additional cases where the joint distributions are known.

Remark 1 *While the asymptotic dependence index is defined for identically distributed random variables and the same threshold* x*, it can easily be extended to cases of non-identically distributed random variables by transformation of variables to a common distribution. See Lemma 14 (in Appendix section) for a sufficient condition.*

Ledford and Tawn (2003), Zhang (2003a), Zhang and Huang (2006) extend the definition of asymptotic dependence between two random variables to lag-k asymptotic dependence of sequences of random variables with identical marginal distribution. The definition of lag-k asymptotic dependence for sequences of random variables is given below.

Definition 2 *Suppose* $\{X_{1d},\ X_{2d},\ \ldots,\ X_{nd},\ d = 1, \ldots, D\}$ *is a D-dimensional multivariate time series with identical marginal distribution. If*

$$\begin{aligned} \lambda_{d_1 d'_{k+1}} &= \lim_{x \to x_F} \mathrm{P}(X_{k+1,d'} > x | X_{1d} > x) > 0, \\ &\lim_{x \to x_F} \mathrm{P}(X_{k+j,d'} > x | X_{1d} > x) = 0,\ j > 1, \end{aligned} \tag{1.3}$$

where $x_F = \sup\{x \in \mathbb{R} :\ \mathrm{P}(X_{1d} \leq x) < 1\}$*, we call the dth series is maximal lag-k tail dependent on the* d'*th series.* $\lambda_{d_1 d'_{i+1}}$ *is called the lag-i tail dependence index of the dth series on the* d'*th series. When* $d = d'$*,* $\lambda_{d_1 d'_{i+1}}$ *is the lag-i tail dependence index within the dth series. When* $i = 0$*,* $\lambda_{d_1 d'_{i+1}}$ *is the tail dependence index between* X_{1d} *and* $X_{1d'}$*. Here* $i = 0, \ldots, k$*.*

1.3 Coefficient of tail dependence

For a broad range of joint distributions, Ledford and Tawn (1996, 1997) consider the following model:

$$\mathrm{P}(X_1 > x,\ X_2 > x) \sim \mathrm{L}\Big(\frac{1}{\mathrm{P}(X_1 > x)}\Big)\mathrm{P}(X_1 > x)^{1/\eta} \qquad \text{as } x \to x_{_F}, \tag{1.4}$$

where L is a slowly varying function, i.e. $\mathrm{L}(tx)/\mathrm{L}(x) \to 1$ as $x \to \infty$ for any fixed $t > 0$, and $\eta \in (0, 1]$ is a constant. Using their terminology, the η value effectively determines the decay rate of the joint bivariate survival function evaluated at the same large x, and η is termed as the coefficient of tail dependence. Two marginal variables are called positively associated when $1/2 < \eta \leq 1$; nearly independent when $\eta = 1/2$; and negatively associated when $0 < \eta < 1/2$. Many coefficients of tail dependence for different joint distributions have been calculated and presented in Heffernan (2000).

Equation (1.4) can be expressed as

$$\mathrm{P}(X_2 > x | X_1 > x) \sim \mathrm{L}\Big(\frac{1}{\mathrm{P}(X_1 > x)}\Big)\mathrm{P}(X_1 > x)^{1/\eta - 1} \qquad \text{as } x \to x_{_F}, \tag{1.5}$$

which shows how λ changes with η. It is easy to see that the two variables X_1 and X_2 are asymptotically dependent when $\eta = 1$ and $\mathrm{L}(x) \nrightarrow 0$ as $x \to \infty$, and are asymptotically independent otherwise.

1.4 The extremal index

Notice that both previous two concepts are used to measure the extremal dependence between two random variables. In reality, extreme events often tend to occur in clusters, and hence a different quantity is needed to measure clustered extremal dependence.

Suppose now $\{X_i, i = 1, 2, \ldots,\}$ is a stationary sequence with a continuous marginal distribution function $F(x)$ and $\{\widehat{X}_i, i = 1, 2, \ldots,\}$ is the so-called associated sequence of i.i.d. random variables with the same marginal distribution function F. The maximum is defined by $M_n = \max\{X_1, \cdots, X_n\}$, while $\widehat{M}_n = \max\{\widehat{X}_1, \cdots, \widehat{X}_n\}$. The limiting distribution of M_n can be related to the limiting distribution of $\widehat{M}_n$ via a quantity θ defined below.

If for every $\tau > 0$ there exists a sequence of thresholds $\{u_n\}$ such that

$$\mathrm{P}\{\widehat{M}_n \leq u_n\} \to e^{-\tau}, \text{ as } n \to \infty \tag{1.6}$$

and under long range dependence conditions $D(u_n),\ D'(u_n)$ of Leadbetter (1983),

$$\mathrm{P}\{M_n \leq u_n\} \to e^{-\theta\tau}, \text{ as } n \to \infty, \tag{1.7}$$

then θ is called the *extremal index* of the sequence $\{X_n\}$.

The value of $1/\theta$ is interpreted as the mean number of exceedances of a threshold per independent cluster as the threshold tends to the upper endpoint of this distribution. When $\theta = 0$, it corresponds to a strong dependence (infinite cluster sizes) but not so strong that all the values can be the same. While $\theta = 1$ is a form of asymptotic independence of extremes, it does not mean that the original sequence is independent.

If (1.7) holds for some τ and corresponding $\{u_n\}$, then it holds for all τ' (equal or not equal to τ) and its corresponding $\{u'_n\}$. Estimators of the extremal index have been proposed by Leadbetter, Weissman, de Haan, and Rootzén (1989), Nandagopalan (1990), Hsing (1993). Smith and Weissman (1994) gave a review of estimating the extremal index and proposed two estimating methods, i.e., blocks method and runs method. Other references include Chapter 8 in Embrechts et al. (1997). Ferro and Segers (2003), and Laurini and Tawn (2003) are more recent references concerning the estimation of the extremal index, among others.

Suppose now $\{\mathbf{X}_i = (X_{i1}, \cdots, X_{iD}), i = 1, 2, ...\}$ is a D-dimensional stationary stochastic processes with distribution function F and marginals F_d, and $\{\mathbf{X}_i\}$ satisfies some long range dependence conditions such as the mixing condition $\Delta(u_n(\boldsymbol{\tau}))$ of Nandagopalan (1994), where $\boldsymbol{\tau} \in T = (0,1)^D \backslash \{\mathbf{1}\}$, $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^D$, or a slightly weaker condition $D(u_n)$ of Hsing (1989). Also let $\{\widehat{\mathbf{X}}_i\}$ be the associated sequence of i.i.d. random vectors having the same distribution function F. $\mathbf{M}_n$ and $\widehat{\mathbf{M}}_n$ are both pointwise maxima of $\{\mathbf{X}_i,\ i = 1, \ldots, n\}$ and $\{\widehat{\mathbf{X}}_i,\ i = 1, \ldots, n\}$ respectively. Suppose

$$\begin{aligned} \lim_{n\to\infty} \mathrm{P}\{M_{n1} \le u_{n1}, \ldots, M_{nD} \le u_{nD}\} &= H(\boldsymbol{\tau}) \\ \lim_{n\to\infty} \mathrm{P}\{\widehat{M}_{n1} \le u_{n1}, \ldots, \widehat{M}_{nD} \le u_{nD}\} &= \widehat{H}(\boldsymbol{\tau}) \end{aligned} \tag{1.8}$$

both exist and are nonzero. The *multivariate extremal index* is defined by

$$H(\boldsymbol{\tau}) = \widehat{H}(\boldsymbol{\tau})^{\theta(\boldsymbol{\tau})} \tag{1.9}$$

where $\theta(\boldsymbol{\tau})$ satisfies

(i) $0 \le \theta(\boldsymbol{\tau}) \le 1$ for all $\boldsymbol{\tau}$,

(ii) $\theta(0, \ldots, 0, \tau_d, 0, \ldots, 0) = \theta_d$ for $\tau_d > 0$, where θ_d is the extremal index of the d^{th} component process.

(iii) $\theta(c\boldsymbol{\tau}) = \theta(\boldsymbol{\tau})$ for all $c > 0$ (Theorem 1.1 of Nandagopalan 1994).

Estimators of the multivariate extremal index have been proposed by Nandagopalan (1994), Weissman (1994), and Smith and Weissman (1996), Martins and Ferreira (2005), among others.

1.5 About the paper

Throughout the rest of the paper, for notational convenience, we denote:

1. $\lambda_{dd'}$ ($\eta_{dd'}$) as the asymptotic dependence index (the coefficient of tail dependence) between the dth component variable and the d'th component variable,

2. λ_{d_k} as the lag k asymptotic dependence index within the dth sequence,

3. and $\lambda_{dd'_k}$ as the lag k asymptotic dependence index cross the dth sequence and the d'th sequence.

In Section 2, we study M4 processes with max-shock variables which have generalized extreme value (GEV) distribution. The processes are asymptotically dependent. In Section 3, we introduce a generalization of M4 class which are asymptotically independent. Simulation examples are presented in Section 4. Discussions are addressed in Section 5. Section 6 are detailed proofs of the main results in Sections 2 and 3.

2 M4 processes with max-shock variables being GEV

2.1 A generalization of M4

In this section, we replace Z_{li} in (1.1) by independent GEV shock variables. We have

$$Y_{id} = \max_{l} \max_{k} a^{-1}_{l,k,d} W_{l,i-k}, \quad d = 1, \ldots, D, \ -\infty < i < \infty, \tag{2.1}$$

for $\{a_{l,k,d} > 0, l \geq 1, -\infty < k < \infty\}$, and $\{W_{lk}, l \geq 1, -\infty < k < \infty\}$ being an array of independent GEV shock variables which have a unified distribution form

$$H(x; \mu, \sigma, \xi) = \exp\Big\{ - \big[1 + \frac{\xi(x-\mu)}{\sigma}\big]^{-1/\xi}\Big\} \tag{2.2}$$

where $1 + \xi(x-\mu)/\sigma > 0$, $\sigma > 0$ and μ, ξ arbitrary. The case $\xi = 0$ is interpreted as the limit $\xi \to 0$, that is

$$H(x; \mu, \sigma, 0) = \exp\Big\{ - \exp\big[- \frac{(x-\mu)}{\sigma}\big]\Big\}, \tag{2.3}$$

which is known as Type I (Gumbel type) extreme value distribution. Type II (Fréchet type) and Type III (Weibull type) correspond to $\xi > 0$ and $\xi < 0$ respectively.

Remark 2 *It is easy to see that the case of $\xi = 1$, $\mu = 0$, $\sigma = 1$, and $\sum_{lk} a^{-1}_{l,k,d} = 1$, $d = 1, \ldots, D$ in Model (2.1) is Smith and Weissman's M4 model.*

In the extreme value literature, models are often specified for the case of $\mu = 0$ and $\sigma = 1$ due to the fact that a linear transformation of random variable gives the same type of the extreme value distribution. In this paper, we also use this formulation. We discuss different cases in the following sections. Some basic marginal distribution, or joint distributions are shown next.

2.2 Some distributional properties

The marginal distribution of Y_{id} is

$$\mathrm{P}(Y_{id} < y) = \exp\Big\{ -\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} (1+\xi a_{l,k,d}y)_+^{-1/\xi}\Big\}, \tag{2.4}$$

where $(\Omega)_+$ equals Ω when $\Omega > 0$; equals 0 otherwise. The multivariate joint distribution of $\{Y_{id},\ i = 1, \ldots, r,\ d = 1, \ldots, D\}$ is

$$\begin{aligned}
&\mathrm{P}\{Y_{id} \le y_{id}, 1 \le i \le r, 1 \le d \le D\} \\
&\quad = \mathrm{P}\{W_{l,i-k} \le a_{l,k,d}y_{id} \text{ for } l \ge 1, -\infty < k < \infty, 1 \le i \le r, 1 \le d \le D\} \\
&\quad = \mathrm{P}\{W_{l,m} \le \min_{1-m\le k\le r-m}\ \min_{1\le d\le D} a_{l,k,d}y_{m+k,d}, l \ge 1, -\infty < m < \infty\} \\
&\quad = \exp\Big\{ -\textstyle\sum_{l=1}^{\infty}\sum_{m=-\infty}^{\infty}(1+\xi \displaystyle\min_{1-m\le k\le r-m}\ \min_{1\le d\le D} a_{l,k,d}y_{m+k,d})_+^{-1/\xi}\Big\}
\end{aligned} \tag{2.5}$$

for appropriate y_{id} values. Particularly for bivariate random variables, we have

$$\mathrm{P}\{Y_{id} \le y,\ Y_{id'} \le y\} = \exp\Big\{ -\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} [1+\xi\min(a_{l,k,d},\ a_{l,k,d'})y]_+^{-1/\xi}\Big\}, \tag{2.6}$$

and

$$\mathrm{P}\{Y_{1d} \le y,\ Y_{rd} \le y\} = \exp\Big\{ -\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} [1+\xi\min(a_{l,k,d},\ a_{l,k+r-1,d})y]_+^{-1/\xi}\Big\}. \tag{2.7}$$

Notice that both parameters (the asymptotic dependence index, the coefficient of tail dependence) are invariant under marginal transformation as long as both original random variables are identically distributed. If two random variables are not identically distributed, special care needs to be taken. In the case of the tail probabilities of two random variables being asymptotically equal, the asymptotic dependence index is invariant under marginal transformation; see Lemma 14 (in Appendix section).

In the following sections, we derive expressions for the asymptotic dependence index, the coefficient of tail dependence and the extremal index for the usual three cases $\xi > 0$, $\xi = 0$, and $\xi < 0$.

One of the following two conditions in deriving asymptotic dependence indexes are needed in order to apply Lemma 14.

C1 Suppose all moving coefficients $a_{l,k,d}$ satisfy

$$\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} a_{l,k,1}^{-1/\xi} = \sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} a_{l,k,d'}^{-1/\xi} < \infty, \text{ for all } d' = 2, \ldots, D. \tag{2.8}$$

C2 Suppose there are numbers $a^* > 0$ and n^* such that

$$\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} a_{l,k,d}^{-1} < \infty,\ a^* = \min_{l}\min_{k} a_{l,k,d},\ n^* = \sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} \mathbf{1}_{(a_{l,k,d}=a^*)}, \text{ for all } d = 1, \ldots, D, \tag{2.9}$$

where $\mathbf{1}_{(.)}$ is an indicator function.

Condition **C1** is particularly for the case of $\xi > 0$, while Condition **C2** is for both cases of $\xi = 0$ and $\xi < 0$. We will show in the subsequent sections that under each of these conditions, the tail probabilities of two random variables Y_{id} and $Y_{id'}$ are asymptotically equal.

2.3 Case $\xi > 0$

With the established notations, the asymptotic dependence indexes of case $\xi > 0$ are presented in the following equations.

Result 3 *Under (2.1) and the condition* **C1**, *we have*

$$\lambda_{dd'_r} = \frac{2\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} a_{l,k,d}^{-1/\xi} - \sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} \min(a_{l,k,d},\ a_{l,k+r,d'})^{-1/\xi}}{\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} a_{l,k,d}^{-1/\xi}}. \tag{2.10}$$

In general $\lambda_{dd'}$ is greater than zero. It is zero when the minimum of every pair $(a_{l,k,d},\ a_{l,k,d'})$ is finite for all $l,\ k$, but the maximum of that pair is infinity. This exceptional case corresponds to the independence case. It is obvious that coefficient of tail dependence $\eta_{dd'}$ is either 1 or 1/2 depending on the value of $\lambda_{dd'}$.

From (2.10), we see that as long as the maximum of $(a_{l,k,d},\ a_{l,k+r,d})$ for some $l,\ m$ is finite, two lag-r variables are asymptotically dependent.

From the above asymptotic dependence indexes, one can immediately see that random variables Y_{id} defined by (2.1) are either independent random variables or asymptotically dependent random variables.

Result 4 *Under Model (2.1) and the $\Delta(u_n(\boldsymbol{\tau}))$ condition, the extremal index is given by*

$$\theta(\boldsymbol{\tau}) = \frac{\sum_{l=1}^{\infty} \max_k \max_d a_{l,k,d}^{-1/\xi}\tau_d}{\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} \max_d a_{l,k,d}^{-1/\xi}\tau_d}. \tag{2.11}$$

2.4 Case $\xi = 0$

The asymptotic dependence indexes of M4 processes with the shock variables being Gumbel type is presented in the following equations.

Result 5 *Under (2.1) and the condition* **C2**, *we have*

$$\lambda_{dd'_r} = \frac{2n^* - \sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} \mathbf{1}_{\left\{\min(a_{l,k,d},\ a_{l,k+r,d'})=a^*\right\}}}{n^*}. \tag{2.12}$$

Notice that when the value of $\lambda_{dd'}$ is greater than 0, the corresponding $\eta_{dd'}$ value is 1.

Also notice that if there are no pairs $(a_{l,k,d},\ a_{l,k,d'})$ equal to a^*, then Y_{id} and $Y_{id'}$ are asymptotically independent. To derive the formula for η parameter when Y_{id} and $Y_{id'}$ are asymptotically independent, it is more convenient to use the following fact for Gumbel type random variable Y_t:

$$\mathrm{P}(Y_t > x) = 1 - e^{-e^{-\alpha x}} \sim e^{-\alpha x}, \quad \text{as } x \to \infty, \text{ and } \alpha > 0. \tag{2.13}$$

Property (2.13) suggests the tail probability of a Gumbel tail may be approximated by an exponential tail probability. We use this fact to introduce multivariate maxima of moving maxima processes based on exponential random variables in Sections 3.2. The formula for the η parameter when the moving variables are Gumbel type shares the same form when the moving variables are unit exponential. We shall derive the formula for η in Section 3.2.

From (2.12), we see that when the values of $(a_{l,k,d},\ a_{l,k+r,d})$ are both equal to a^* for some m value(s), we observe the lag-r asymptotic dependence in the dth sequence.

From (2.12), we see that we can actually obtain asymptotically independent multivariate stationary processes by imposing a condition that there are no identical pairs $(a_{l,k,d},\ a_{l,k+r,d'})$, $d = 1, \ldots, D$, $d' = 1, \ldots, D$, $r = 0, 1, 2, \ldots$, in which they are equal to a^*. When D is very large, it may be hard for a practically feasible model to satisfy this condition since it requires that l varies in a very large range. A modification of (2.1) will solve this practical issue. In Section 3, we address the modification.

Result 6 *Under Model (2.1) and the $\Delta(u_n(\boldsymbol{\tau}))$ condition, the extremal index is given by*

$$\theta(\boldsymbol{\tau}) = \frac{\sum_{l=1}^{\infty} \mathbf{1}_{\left(\min\limits_{-\infty<k<\infty} \min_d a_{l,k,d} = a^*\right)} \exp(-\min\limits_{-\infty<k<\infty} \min_d a_{l,k,d}/\tau_d)}{\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \mathbf{1}_{(\min_d a_{l,k,d} = a^*)} \exp(-\min_d a_{l,k,d}/\tau_d)}. \tag{2.14}$$

2.5 Case $\xi < 0$

Suppose condition **C2** holds. Let a^{**} be the second smallest of all $\{a_{l,k,d},\ l \geq 1,\ -\infty < k < \infty\}$. Then for $\frac{-1}{a^{**}\xi} < y < \frac{-1}{a^*\xi}$, from (2.4)-(2.7), we have

$$\mathrm{P}(Y_{id} < y) = \exp\Big\{ -(1+\xi a^* y)^{-1/\xi} \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \mathbf{1}_{\{a_{l,k,d} = a^*\}} \Big\},$$

$$\mathrm{P}\{Y_{id} \leq y,\ Y_{id'} \leq y\} = \exp\Big\{ -(1+\xi a^* y)^{-1/\xi} \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \mathbf{1}_{\{\min(a_{l,k,d},\ a_{l,k,d'}) = a^*\}} \Big\},$$

$$\mathrm{P}\{Y_{1d} \leq y,\ Y_{rd} \leq y\} = \exp\Big\{ -(1+\xi a^* y)^{-1/\xi} \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \mathbf{1}_{\{\min(a_{l,k,d},\ a_{l,k+r-1,d}) = a^*\}} \Big\}.$$

These three equations will be used in the derivation of asymptotic dependence indexes and coefficients of tail dependence, and the following **Result 7** is obtained.

Result 7 *Under (2.1) and and the condition* **C2***, we have the same results as in* **Result 5***.*

As long as the asymptotic dependence indexes are not zero, we have the corresponding η being 1.

When $\lambda_{dd'} = 0$, we have $\lim_{x \to \frac{-1}{a^*\xi}} P(Y_{1d} > x, Y_{1d'} > x)/P(Y_{1d} > x)^2 = 1$, therefore, the coefficient of tail dependence is $\eta_{dd'} = 1/2$. Similarly, $\eta_{dd'_r} = 1/2$ for all r.

Result 8 *Under Model (2.1) and the $\Delta(u_n(\boldsymbol{\tau}))$ condition,, the extremal index is given by*

$$\theta(\boldsymbol{\tau}) = \frac{\sum_{l=1}^{\infty}(1 + \xi \min_k \min_d a_{l,k,d}/\tau_d)^{-1/\xi}\mathbf{1}_{\{\min_k \min_d a_{l,k,d} = a^*\}}}{\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}(1 + \xi \min_d a_{l,k,d}/\tau_d)^{-1/\xi}\mathbf{1}_{\{\min_d a_{l,k,d} = a^*\}}}. \tag{2.15}$$

3 Asymptotic Independent M4 Processes

From the previous sections, we saw that in order to get asymptotic independence, we had to assume those "paired" coefficients are not identical, and not equal to a^*, the minimum of $a_{l,k,d}$, as defined in Condition **C2**. We propose a revision of M4 process such that that restriction is no longer needed. The new model is as follows:

$$Y_{id} = \max(U_{id}^{1/\alpha},\ \max_l\ \max_k a_{l,k,d}^{-1} W_{l,i-k}), \quad d = 1, \ldots, D, \tag{3.1}$$

where $\alpha > 0$, $a_{l,k,d} > 0$, $\{W_{li}, l \geq 1, -\infty < i < \infty\}$ are an array of independent positive random variables; $\{U_{id}, -\infty < i < \infty,\ d = 1, \ldots, D\}$ are an array of independent positive random variables, and they are independent of W_{li}. These max-shock random variables are identically distributed.

In this section, we consider two underlying distribution assumptions: the first is based on unit Fréchet variables; the second is based on unit exponential variables. The reason to choose unit Fréchet variables is due to that it is a max-stable distribution with positive support. The reason to choose unit exponential is that its tail is very close to the tail of Gumbel distribution, and we know the reciprocal of a unit exponential random variable is a unit Fréchet variable.

3.1 Extended M4 processes with max-shock variable being unit Fréchet

In this case, we make the following assumption:

$$\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} a_{l,k,d}^{-1} = 1,\ d = 1, \ldots, D. \tag{3.2}$$

Then, some basic marginal distribution or joint distributions are derived as follows:

$$P(Y_{id} \leq y) = \exp(-\frac{1}{y^\alpha} - \frac{1}{y}), \tag{3.3}$$

$$P(Y_{id} \leq y_{id}, Y_{i+r,d} \leq y_{i+r,d}) = \exp\{-\frac{1}{y_{id}^\alpha} - \frac{1}{y_{i+r,d}^\alpha}\}\exp\Big[-\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\max\{\frac{a_{l,k,d}^{-1}}{y_{id}}, \frac{a_{l,k+r,d}^{-1}}{y_{i+r,d}}\}\Big], \tag{3.4}$$

$$P(Y_{1d} \leq y_{1d}, Y_{1d'} \leq y_{1d'}) = \exp\{-\frac{1}{y_{1d}^\alpha} - \frac{1}{y_{1d'}^\alpha}\}\exp\Big[-\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\max\{\frac{a_{l,k,d}^{-1}}{y_{1d}}, \frac{a_{l,k,d'}^{-1}}{y_{1d'}}\}\Big]. \tag{3.5}$$

These three equations will be used in the derivation of the asymptotic dependence indexes and the coefficients of tail dependence.

Result 9 *Under (3.1) and (3.2), we have*

$$\lambda_{dd'_r} = \begin{cases} 0, & \text{if } \alpha < 1; \\ 2 - \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \max(a_{l,k,d}^{-1},\ a_{l,k+r,d'}^{-1}), & \text{if } \alpha \geq 1. \end{cases} \tag{3.6}$$

Result 10 *The coefficients of tail dependence are given by*

$$\eta = \begin{cases} \max(1/2, \alpha), & \text{if } \alpha < 1; \\ 1, & \text{if } \alpha \geq 1. \end{cases} \tag{3.7}$$

Notice that here η values do not depend on the coefficients $a_{l,k,d}$. In the next section, we introduce a model with short tail max-shock variables which result in different η values depending on the coefficients $a_{l,k,d}$.

Under the $\Delta(u_n(\boldsymbol{\tau}))$ condition, by taking $y_d = x_d n^{1/\alpha}$ when $\alpha < 1$, and $y_d = x_d n$ when $\alpha \geq 1$, we can easily show that the extremal index is 1 when $\alpha \geq 1$; otherwise it is the same as the extremal index derived for the case $\xi > 0$ in Section 2.3.

3.2 Extended M4 process with max-shock variable being unit exponential

We assume now W_{li} and U_{id} in (3.1) are unit exponential random variables. We also assume that Condition **C2** holds, and all $a_{l,k,d} > 1$.

The marginal distribution is

$$\mathrm{P}(Y_{id} < y) = (1 - e^{-y^{\alpha}}) \prod_{l=1}^{\infty} \prod_{k=-\infty}^{\infty} (1 - e^{-a_{l,k,d} y}). \tag{3.8}$$

Some joint distributions of Y_{id}s are shown below:

$$\begin{aligned} &\mathrm{P}\{Y_{id} \leq y_{id}, 1 \leq i \leq r, 1 \leq d \leq D\} \\ &\quad = (1 - e^{-y^{\alpha}})^{rD} \mathrm{P}\{E_{l,i-k} \leq a_{l,k,d} y_{id} \text{ for } l \geq 1, -\infty < k < \infty, 1 \leq i \leq r, 1 \leq d \leq D\} \\ &\quad = (1 - e^{-y^{\alpha}})^{rD} \mathrm{P}\{E_{l,m} \leq \min_{1-m \leq k \leq r-m} \min_{1 \leq d \leq D} a_{l,k,d} y_{m+k,d}, l \geq 1, -\infty < m < \infty\} \\ &\quad = (1 - e^{-y^{\alpha}})^{rD} \textstyle\prod_{l=1}^{\infty} \prod_{k=-\infty}^{\infty} (1 - e^{-\min_{1-m \leq k \leq r-m} \min_{1 \leq d \leq D} a_{l,k,d} y_{m+k,d}}), \end{aligned} \tag{3.9}$$

$$\mathrm{P}\{Y_{id} \leq y,\ Y_{id'} \leq y\} = (1 - e^{-y^{\alpha}})^2 \prod_{l=1}^{\infty} \prod_{k=-\infty}^{\infty} (1 - e^{-\min(a_{l,k,d},\ a_{l,k,d'}) y}), \tag{3.10}$$

and

$$\mathrm{P}\{Y_{1d} \leq y,\ Y_{rd} \leq y\} = (1 - e^{-y^{\alpha}})^2 \prod_{l=1}^{\infty} \prod_{k=-\infty}^{\infty} (1 - e^{-\min(a_{l,k,d},\ a_{l,k+r-1,d}) y}). \tag{3.11}$$

The asymptotic dependence indexes are derived as follows:

Result 11 *Under (3.1), (2.9) and* $a_{l,k,d} > 1$*, we have*

$$\lambda_{dd'_r} = \begin{cases} 0, & \text{if } \alpha \leq 1; \\ \frac{2n^* - \sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} \mathbf{1}_{\min(a_{l,k,d}, a_{l,k+r,d'})=a^*}}{n^*}, & \text{if } \alpha > 1. \end{cases} \tag{3.12}$$

Notice that when $\alpha \leq 1$, we always get asymptotic independence. When $\alpha > 1$ and no pairs of $(a_{l,k,1},\ a_{l,2-m,1})$ (and similarly for other cases) are identically equal to a^*, then we get asymptotic independence, otherwise, asymptotic dependence. Also notice that the α values result in different asymptotic dependence structures for the two different type max-shock variables.

In order to compute the coefficients of tail dependence, we first establish the following equation:

$$1 - \sum_{s=d,d'} \prod_{l=1}^{\infty} \prod_{k=-\infty}^{\infty} (1 - e^{-a_{l,k,s}y}) + \prod_{l=1}^{\infty} \prod_{k=-\infty}^{\infty} (1 - e^{-\min(a_{l,k,d},\ a_{l,k+r,d'})y}) = \sum_{m=1}^{\infty} g_m^{(r)} e^{-b_m^{(r)} y}, \tag{3.13}$$

where $b_1^{(r)} < b_2^{(r)} < \ldots$, and $g_m^{(r)}$ are non-zero constants.

Result 12 *Under (3.1) and (3.13), we have*

$$\eta_{dd'_r} = \begin{cases} 1/2, & \text{if } \alpha \leq 1; \\ a^*/b_1^{(r)}, & \text{if } \alpha > 1, \text{ and no pairs of } (a_{l,k,d},\ a_{l,k+r,d'}) \text{ are identically equal to } a^*. \end{cases} \tag{3.14}$$

Notice that other cases can be illustrated similarly. Examples regarding finding η values are given in Section 4.

To derive the extremal index, we first present the following lemma for a family of Weibull distributions.

Lemma 13 *Let* $G(x) = 1 - e^{-\beta x^{\alpha}},\ \alpha > 0,\ \beta > 0,\ x > 0$*, then*

$$n[1 - G(u_n)] \to e^{-x} \tag{3.15}$$

where $u_n = a_n x + b_n$*,* $a_n = \beta^{-1/\alpha}[\log(n)]^{1/\alpha - 1}$*, and* $b_n = \beta^{-1/\alpha}[\log(n)]^{1/\alpha}$*.*

Under the $\Delta(u_n(\boldsymbol{\tau}))$ condition, using Lemma 13, one immediately sees that $\theta(\tau) = 1$ when $\alpha \leq 1$. When $\alpha > 1$, the extremal index is the same as case $\{\xi = 0\}$ in Section 2.4, i.e.

$$\theta(\boldsymbol{\tau}) = \frac{\sum_{l=1}^{\infty} \mathbf{1}_{\left(\min\limits_{-\infty<k<\infty} \min_d a_{l,k,d} = a^*\right)} \exp(-\min\limits_{-\infty<k<\infty} \min_d a_{l,k,d}/\tau_d)}{\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} \mathbf{1}_{(\min_d a_{l,k,d}=a^*)} \exp(-\min_d a_{l,k,d}/\tau_d)}. \tag{3.16}$$

4 Simulation examples

Example 4.1 *Consider the following two different extended M4 processes:*

$$Y_{id} = \max\{a_{1d}^{-1}W_{i-1},\ a_{2d}^{-1}W_{i-2},\ a_{3d}^{-1}W_{i-3}\}, \quad d = 1, 2; \tag{4.1}$$

and

$$Y'_{id} = \max\{U_{id}^{1/\alpha},\ \max(a_{1d}^{-1}W_{i-1},\ a_{2d}^{-1}W_{i-2})\}, \quad d = 1, 2. \tag{4.2}$$

When we apply Model (2.1), max-shock variables are GEV random variables, and we consider $\xi = 1,\ \xi = 0,\ \xi = -1$ *in both models (4.1) and (4.2). When we apply Model (3.1), they are unit Fréchet or unit exponential random variables. In each process, we consider three cases:*

$$SC1: \ (\alpha;\ a_{11},\ a_{21},\ a_{31};\ a_{12},\ a_{22},\ a_{32}) = (1/3;\ 2,\ 5,\ 10;\ 2,\ 10,\ 5),$$

$$SC2: \ (\alpha;\ a_{11},\ a_{21},\ a_{31};\ a_{12},\ a_{22},\ a_{32}) = (2/3;\ 2,\ 3,\ 10;\ 2,\ 10,\ 3),$$

and

$$SC3: \ (\alpha;\ a_{11},\ a_{21};\ a_{12},\ a_{22}) = (1.5;\ 2,\ 10;\ 10,\ 2).$$

The other $a_{l,k,d}$*s take values infinity in each case. Values for* α *are not used in Model (2.1).*

The computed values of asymptotic dependence indexes, coefficients of tail dependence for different models (4.1) and (4.2) and max-moving coefficients (SC1, SC2, SC3) are listed in Table 1. The simulated bivariate processes of different models and max-moving coefficients are plotted in Figure 1 at Gumbel scales.

From Table 1, we can see that M4 processes with GEV max-shock variables and extended M4 classes have flexibilities to model nearly independent, positive dependent, and asymptotic dependent variables. From Figure 1, these two dimensional scatter plots show various shapes between two random variables, and hence they suggest that these models are suitable for a wide range of dependence structures.

5 Discussion

In this paper, we have demonstrated that M4 processes with GEV max-shock variables can model nearly independent, positive dependent, and asymptotic dependent variables. Particularly, when $\xi > 0$, (2.1) is either asymptotically dependent or independent; when $\xi < 0$, (2.1) is either asymptotic dependent or nearly independent; and when $\xi = 0$, (2.1) also models positive dependencies. Extended M4 classes also have the properties of modeling near dependence, positive dependence, and asymptotic dependence. These models give model builders more flexibilities in real data modeling. They may lead to a new research field in extreme value theory and multivariate time series modeling. Like other time series models, model selection and parameter estimation are two important tasks. Constructions of parameter estimators in these models and their applications to real data are our future research direction.

Parameter		Model Specification				
		$\xi = 1$	$\xi = 0$	$\xi = -1$	(4.2) Fréchet	(4.2) exponential
SC1	λ_{12}	.875	1.0	1.0	0	0
	η_{12}	1.0	1.0	1.0	.5	.5
	λ_{1_1}	.25	0	0	0	0
	η_{1_1}	1.0	.5	.5	.5	.5
	λ_{2_1}	.375	0	0	0	0
	η_{2_1}	1.0	.5	.5	.5	.5
	λ_{12_1}	.375	0	0	0	0
	η_{12_1}	1.0	.5	.5	.5	.5
	λ_{21_1}	.375	0	0	0	0
	η_{21_1}	1.0	.5	.5	.5	.5
SC2	λ_{12}	.75	1.0	1.0	0	1.0
	η_{12}	1.0	1.0	1.0	.6667	1.0
	λ_{1_1}	.2143	0	0	.0	0
	η_{1_1}	1.0	.5	.5	.6667	.5
	λ_{2_1}	.4643	0	0	0	0
	η_{2_1}	1.0	.6667	.5	.6667	.6667
	λ_{12_1}	.4643	0	0	0	0
	η_{12_1}	1.0	.6667	.5	.6667	.6667
	λ_{21_1}	.4643	0	0	0	0
	η_{21_1}	1.0	.5	.5	.6667	.5
SC3	λ_{12}	.3333	0	0	.3333	0
	η_{12}	1.0	.5	.5	1.0	.5
	λ_{1_1}	.1667	0	0	.1667	0
	η_{1_1}	1.0	.5	.5	1.0	.5
	λ_{2_1}	.1667	0	0	.1667	0
	η_{2_1}	1.0	.5	.5	1.0	.5
	λ_{12_1}	.8333	1.0	1.0	.8333	1.0
	η_{12_1}	1.0	1.0	1.0	1.0	1.0
	λ_{21_1}	.1667	0	0	.1667	0
	η_{21_1}	1.0	.5	.5	1.0	.5

Table 1: Asymptotic dependence indexes, coefficients of tail dependence for different models and max-moving coefficients.

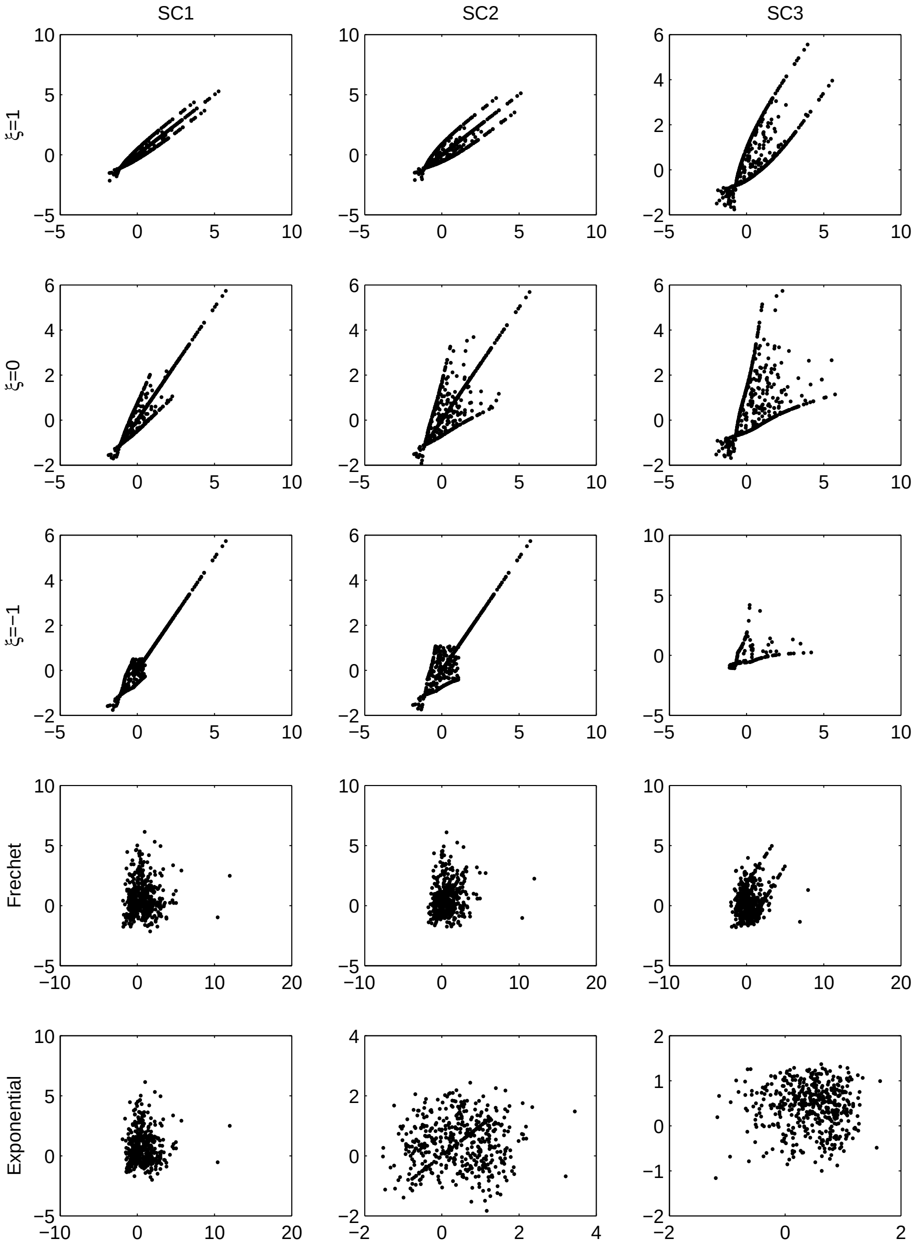

Figure 1: *Comparison between cross-sectional dependencies for different models and max-moving coefficients in Example 4.1. The data in the figure has been transformed to Gumbel scales.*

6 Appendix

The following lemma tells that when we compute the asymptotic dependence index, and coefficient of tail dependence between two random variables, we do not need them to have the same marginal distribution.

Lemma 14 *Suppose X and Y satisfy $\mathrm{P}(X > x)/\mathrm{P}(Y > x) \to 1$ as x tends to infinity. Y' is the marginally transformed random variable of Y, i.e. $Y' = G(Y)$ for some increasing monotone function G; and Y' has the same distribution as X has. Then*

$$\lim_{x\to\infty} \mathrm{P}(Y > x|X > x) = \lim_{u\to\infty} \mathrm{P}(Y' > x|X > x) \tag{6.1}$$

as long as one of the above two limits exists.

Proof . As x tends to infinity, we first have

$$\frac{\mathrm{P}(X > x)}{\mathrm{P}(Y > x)} \to 1, \; \frac{\mathrm{P}(X > x)}{\mathrm{P}(Y' > x)} = 1$$

which imply

$$\frac{\mathrm{P}(Y > x)}{\mathrm{P}(Y' > x)} \to 1$$

which implies

$$\frac{\mathrm{P}(Y > x)}{\mathrm{P}[Y > G^{-1}(x)]} \to 1, \; \frac{\mathrm{P}[Y > \min\{x, G^{-1}(x)\}]}{\mathrm{P}[Y > G^{-1}(x)]} \to 1, \; \frac{\mathrm{P}[Y > \max\{x, G^{-1}(x)\}]}{\mathrm{P}[Y > G^{-1}(x)]} \to 1.$$

We have

$$\begin{aligned} &\frac{\mathrm{P}[X > \max\{x, G^{-1}(x)\}, \; Y > \max\{x, G^{-1}(x)\}]}{\mathrm{P}(Y' > x)} \\ &\quad \le \frac{\mathrm{P}(X > x, \; Y' > x)}{\mathrm{P}(Y' > x)} = \frac{\mathrm{P}[X > x, \; Y > G^{-1}(x)]}{\mathrm{P}(Y' > x)} \\ &\quad\quad \le \frac{\mathrm{P}[X > \min\{x, G^{-1}(x)\}, \; Y > \min\{x, G^{-1}(x)\}]}{\mathrm{P}(Y' > x)}, \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} &\frac{\mathrm{P}[X > \max\{x, G^{-1}(x)\}, \; Y > \max\{x, G^{-1}(x)\}]}{\mathrm{P}(Y' > x)} \\ &\quad = \frac{\mathrm{P}[X > \max\{x, G^{-1}(x)\}, \; Y > \max\{x, G^{-1}(x)\}]}{\mathrm{P}[X > \max\{x, G^{-1}(x)\}]} * \frac{\mathrm{P}[X > \max\{x, G^{-1}(x)\}]}{\mathrm{P}[Y > \max\{x, G^{-1}(x)\}]} \\ &\quad\quad * \frac{\mathrm{P}[Y > \max\{x, G^{-1}(x)\}]}{\mathrm{P}[Y > G^{-1}(x)]}, \end{aligned} \tag{6.3}$$

$$\frac{\mathrm{P}[X > \min\{x, G^{-1}(x)\},\ Y > \min\{x, G^{-1}(x)\}]}{\mathrm{P}(Y' > x)}$$

$$= \frac{\mathrm{P}[X > \min\{x, G^{-1}(x)\},\ Y > \min\{x, G^{-1}(x)\}]}{\mathrm{P}[X > \min\{x, G^{-1}(x)\}]} * \frac{\mathrm{P}[X > \min\{x, G^{-1}(x)\}]}{\mathrm{P}[Y > \min\{x, G^{-1}(x)\}]} * \frac{\mathrm{P}[Y > \min\{x, G^{-1}(x)\}]}{\mathrm{P}[Y > G^{-1}(x)]} \tag{6.4}$$

Taking $x \to \infty$ in Equations (6.2)-(6.4), we get (6.1). □

Proof of **Result** 3. Since

$$\lim_{x\to\infty} \frac{P(Y_{1d} > x)}{P(Y_{1d'} > x)} = \lim_{x\to\infty} \frac{1 - \exp\Big\{ -\sum_l \sum_k (1+\xi a_{l,k,d} x)^{-1/\xi}\Big\}}{1 - \exp\Big\{ -\sum_l \sum_k (1+\xi a_{l,k,d'} x)^{-1/\xi}\Big\}} = \frac{\sum_l \sum_k a_{l,k,d}^{-1/\xi}}{\sum_l \sum_k a_{l,k,d'}^{-1/\xi}},$$

so by Condition (2.8), we can apply Lemma 14.

Note that

$$\frac{P(Y_{1d} > x, Y_{1+r,d'} > x)}{P(Y_{1d} > x)} = \frac{P(Y_{1d} > x) + P(Y_{1+r,d'} > x) - [1 - P(Y_{1d} < x, Y_{1+r,d'} < x)]}{P(Y_{1d} > x).} \tag{6.5}$$

As $x \to \infty$, we have $1 - e^{-x} \sim x$, and by (2.4), we have

$$P(Y_{1d} > x) = 1 - \exp\Big\{ -\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} (1+\xi a_{l,k,d} y)_+^{-1/\xi}\Big\}$$
$$\sim \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} (1+\xi a_{l,k,d} y)_+^{-1/\xi} \sim \xi^{-1/\xi} x^{-1/\xi} \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} a_{l,k,d}^{-1/\xi} \to 0.$$

Replacing $a_{l,k,d}$ by $\min(a_{l,k,d}, a_{l,k+r,d})$ in the above expression, we have

$$1 - P(Y_{1d} < x, Y_{1+r,d'} < x) = 1 - \exp\Big\{ -\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} (1+\xi \min(a_{l,k,d}, a_{l,k+r,d}) x)_+^{-1/\xi}\Big\}$$
$$\sim \xi^{-1/\xi} x^{-1/\xi} \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \min(a_{l,k,d}, a_{l,k+r,d})^{-1/\xi}.$$

Substitution of these results in (6.5), we can immediately get the asymptotic dependence index between these two lagged variables Y_{id} and $Y_{i+r,d'}$ by:

$$\lambda_{dd'_r} = \lim_{x\to\infty} \frac{P(Y_{1d} > x, Y_{1+r,d'} > x)}{P(Y_{1d} > x)} = \frac{2\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} a_{l,k,d}^{-1/\xi} - \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \min(a_{l,k,d},\ a_{l,k+r,d'})^{-1/\xi}}{\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} a_{l,k,d}^{-1/\xi}}.$$

□

Proof of **Result** 4. We denote independent sequence $\{\tilde{Y}_{id},\ i = 1, \ldots, n\}$ as the associated sequence of Y_{id} for each d. Then $\{\tilde{Y}_{id}\}$ has the marginal distribution (2.4), and

$$\mathrm{P}(\tilde{Y}_{id} < y,\ 1 \le i \le n) = \exp\{-n \sum_l \sum_k (1 + \xi y a_{l,k,d})^{-1/\xi}\}. \tag{6.6}$$

Let $y = \xi^{-1} n^{\xi} x$, then

$$\begin{aligned}
\mathrm{P}(\tilde{Y}_{id} < \xi^{-1} n^{\xi} x,\ 1 \le i \le n) &= \exp\{-n \sum_l \sum_k (1 + n^{\xi} x a_{l,k,d})^{-1/\xi}\} \\
&= \exp\{- \sum_l \sum_k (n^{-\xi} + x a_{l,k,d})^{-1/\xi}\} \\
&\to e^{-x^{-1/\xi} (\sum_l \sum_k a_{l,k,d}^{-1/\xi})}.
\end{aligned}$$

Notice that when $\sum_l \sum_k a_{l,k,d}^{-1/\xi} = 1$ and $\xi = 1$, we get Smith and Weissman's M4 processes. For the original univariate sequence Y_{id}, we have

$$\begin{aligned}
\mathrm{P}(Y_{id} < \xi^{-1} n^{\xi} x,\ 1 \le i \le n) &= \exp\{- \sum_l \sum_m (1 + n^{\xi} x \min_{1-m \le k \le n-m} a_{l,k,d})^{-1/\xi}\} \\
&= e^{-x^{-1/\xi} n^{-1} \sum_l \sum_m (n^{-\xi} x + \min_{1-m \le k \le n-m} a_{l,k,d})^{-1/\xi}}.
\end{aligned}$$

For any $\delta > 0$, there is an N such that when $n > N$, $n^{-\xi} x < \delta$, and

$$\Big(\min_{1-m \le k \le n-m} (\delta + a_{l,k,d}) \Big)^{-1/\xi} < \Big(n^{-\xi} x + \min_{1-m \le k \le n-m} a_{l,k,d} \Big)^{-1/\xi} < \Big(\min_{1-m \le k \le n-m} a_{l,k,d} \Big)^{-1/\xi},$$

then by using similar arguments of Theorem 3.1 and Lemma 3.2 of Smith and Weissman (1996), we have

$$\mathrm{P}(Y_{id} < \xi^{-1} n^{\xi} x,\ 1 \le i \le n) \to e^{-x^{-1/\xi} (\sum_l \max_k a_{l,k,d}^{-1/\xi})}$$

which gives the extremal index for the dth sequence as follows:

$$\theta_d = \frac{\sum_l \max_k a_{l,k,d}^{-1/\xi}}{\sum_l \sum_k a_{l,k,d}^{-1/\xi}}.$$

For a D dimensional process, let $\boldsymbol{u}_n = (\xi^{-1} n^{\xi} \tau_1^{-1},\ \ldots, \xi^{-1} n^{\xi} \tau_D^{-1})$, then we have

$$\begin{aligned}
\mathrm{P}\{\tilde{M}_n \le \boldsymbol{u}_n\} &= \exp\Big\{ - \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} (1 + n^{\xi} \min_{1 \le d \le D} a_{l,k,d} \tau_d^{-1})_+^{-1/\xi} \Big\} \\
&\to \exp\Big\{ - \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \max_{1 \le d \le D} a_{l,k,d}^{-1/\xi} \tau_d \Big\} \quad \text{as } n \to \infty,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{P}\{M_n \le \boldsymbol{u}_n\} &= \exp\Big\{ - \sum_{l=1}^{\infty} \sum_{m=-\infty}^{\infty} (1 + n^{\xi} \min_{1-m \le k \le n-m} \min_{1 \le d \le D} a_{l,k,d} \tau_d)_+^{-1/\xi} \Big\} \\
&\to \exp\Big\{ - \sum_{l=1}^{\infty} \max_{-\infty < k < \infty} \max_{1 \le d \le D} a_{l,k,d}^{-1/\xi} \tau_d \Big\} \quad \text{as } n \to \infty.
\end{aligned}$$

These expressions lead to (2.11). □

Proof of **Result** 5. It is easy to check that Lemma 14 can be applied based on Condition (2.9). For each d and $x > 0$, we have $\exp\{-a_{l,k,d}x\} < a_{l,k,d}^{-2}$ when l and k are sufficiently large, and hence $\lim_{x\to\infty}\sum_l\sum_k a_{l,k,d}\exp\{-a_{l,k,d}x\} = 0$, $\lim_{x\to\infty}\sum_l\sum_k a_{l,k,d}\exp\{-(a_{l,k,d}-a^*)x\} = n^*a^*$.

Using L'Hospital's Rule, we have

$$\begin{aligned}
\lambda_{dd'_r} &= \lim_{x\to\infty}\frac{P(Y_{1d} > x, Y_{1+r,d'} > x)}{P(Y_{1d} > x)}\\
&= \lim_{x\to\infty}\frac{1-\sum\limits_{h=d,d'}\exp\Big\{-\sum\limits_l\sum\limits_k\exp(-a_{l,k,h}x)\Big\}+\exp\Big\{-\sum\limits_{l=1}^{\infty}\sum\limits_{k=-\infty}^{\infty}\exp[-\min(a_{l,k,d},\ a_{l,k+r-1,d'})x]\Big\}}{1-\exp\Big\{-\sum_l\sum_k\exp(-a_{l,k,d}x)\Big\}}\\
&= \lim_{x\to\infty}\Big[\frac{\sum\limits_{h=d,d'}\sum\limits_l\sum\limits_k a_{l,k,h}\exp(-a_{l,k,h}x)\exp\Big\{-\sum\limits_l\sum\limits_k\exp(-a_{l,k,h}x)\Big\}}{\sum_l\sum_k a_{l,k,d}\exp(-a_{l,k,d}x)\exp\Big\{-\sum_l\sum_k\exp(-a_{l,k,d}x)\Big\}}\\
&\quad-\frac{\sum\limits_{l=1}^{\infty}\sum\limits_{k=-\infty}^{\infty}\min(a_{l,k,d},\ a_{l,k+r-1,d'})\exp[-\min(a_{l,k,d},\ a_{l,k+r-1,d'})x]}{\sum_l\sum_k a_{l,k,d}\exp(-a_{l,k,d}x)}\\
&\quad*\frac{\exp\Big\{-\sum\limits_{l=1}^{\infty}\sum\limits_{k=-\infty}^{\infty}\exp[-\min(a_{l,k,d},\ a_{l,k+r-1,d'})x]\Big\}}{\exp\Big\{-\sum_l\sum_k\exp(-a_{l,k,d}x)\Big\}}\Big]\\
&= \frac{\sum\limits_{h=d,d'}\sum\limits_l\sum\limits_k a_{l,k,h}\mathbf{1}_{(a_{l,k,h}=a^*)}-\sum\limits_{l=1}^{\infty}\sum\limits_{k=-\infty}^{\infty}\min(a_{l,k,d},\ a_{l,k+r-1,d'})\mathbf{1}_{(\min(a_{l,k,d},\ a_{l,k+r-1,d'})=a^*)}}{\sum_l\sum_k a_{l,k,d}\mathbf{1}_{(a_{l,k,d}=a^*)}}.
\end{aligned}$$

□

Proof of **Result** 6. From (6.6) and $y = x + \log(n^{1/a^*})$, we have

$$\begin{aligned}
\mathrm{P}(\tilde{Y}_{id} < x+\log(n^{1/a^*}),\ 1\le i\le n) &= \exp\{-n\sum_l\sum_k\exp(-ya_{l,k,d})\}\\
&= \exp\{-n\sum_l\sum_k[\exp(-a_{l,k,d}x)\exp\{-a_{l,k,d}\log(n^{1/a^*})\}]\}\\
&= \exp\{-n\sum_l\sum_k n^{-a_{l,k,d}/a^*}[\exp(-a_{l,k,d}x)]\}\\
&\to \exp\big(-\exp[-a^*x+\log\{\sum_l\sum_k\mathbf{1}_{(a_{l,k,d}=a^*)}\}]\big).
\end{aligned}$$

which is in the domain of Gumbel type.

$$\begin{aligned}
\mathrm{P}(Y_{id} < x+\log(n^{1/a^*}),\ 1\le i\le n) &= \exp\big(-\sum_l\sum_m\exp(-\min_{1-m\le k\le n-m}a_{l,k,d}x)\\
&\quad*\exp\{-\min_{1-m\le k\le n-m}a_{l,k,d}\log(n^{1/a^*})\}\big)\\
&\to \exp\big(-\exp[-a^*x+\log\{\sum_l\mathbf{1}_{[\min\limits_{-\infty<k<\infty}a_{l,k,d}=a^*]}\}]\big).
\end{aligned}$$

which gives the extremal index for the dth sequence as follows:

$$\theta_d = \frac{\sum_l \mathbf{1}_{[\min\limits_{-\infty<k<\infty} a_{l,k,d}=a^*]}}{\sum_l \sum_k \mathbf{1}_{(a_{l,k,d}=a^*)}}.$$

From (6.6) and $y_d = x_d + \log(n^{1/a^*}),\ d = 1, \ldots, D$, we have

$$\begin{aligned}
\mathrm{P}(\tilde{Y}_{id} < x_d + &\log(n^{1/a^*}),\ 1 \le i \le n,\ 1 \le d \le D)\\
&= \exp\{-n \sum_l \sum_k \exp(-\min_d a_{l,k,d} y_d)\}\\
&= \exp\{-n \sum_l \sum_k [\exp(-\min_d a_{l,k,d} x_d) \exp\{-\min_d a_{l,k,d} \log(n^{1/a^*})\}]\}\\
&= \exp\{-n \sum_l \sum_k n^{-\min_d a_{l,k,d}/a^*} [\exp(-\min_d a_{l,k,d} x_d)]\}\\
&\to \exp\Big\{-\sum_l \sum_k \mathbf{1}_{(\min_d a_{l,k,d}=a^*)} \exp(-\min_d a_{l,k,d} x_d)\Big\}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{P}(Y_{id} < x_d + &\log(n^{1/a^*}),\ 1 \le i \le n,\ d = 1, \ldots, D)\\
&= \exp\Big(-\sum_l \sum_m \exp(-\min_{1-m\le k\le n-m} \min_d a_{l,k,d} x_d)\\
&\qquad\qquad * \exp\{-\min_{1-m\le k\le n-m} \min_d a_{l,k,d} \log(n^{1/a^*})\}\Big)\\
&\to \exp\Big\{-\sum_l \mathbf{1}_{[\min\limits_{-\infty<k<\infty} \min_d a_{l,k,d}=a^*]} \exp(-\min_{-\infty<k<\infty} \min_d a_{l,k,d} x_d)\Big\}.
\end{aligned}$$

Similar to the proof of Result 4, the extremal index for the multivariate processes is proved to be (2.14). $\square$

Proof of **Result** 7. Since

$$\lim_{x\to\frac{-1}{a^*\xi}} \frac{P(Y_{1d} > x)}{P(Y_{1d'} > x)} = \lim_{x\to\frac{-1}{a^*\xi}} \frac{1-\exp\Big\{-\sum_l\sum_k(1+\xi a_{l,k,d}x)^{-1/\xi}\Big\}}{1-\exp\Big\{-\sum_l\sum_k(1+\xi a_{l,k,d'}x)^{-1/\xi}\Big\}} = \frac{\sum_l\sum_k \mathbf{1}_{\{a_{l,k,d}=a^*\}}}{\sum_l\sum_k \mathbf{1}_{\{a_{l,k,d'}=a^*\}}},$$

so Lemma 14 can be applied.

We have

$$\begin{aligned}
\lambda_{dd'_r} &= \lim_{x\to\frac{-1}{a^*\xi}} \frac{P(Y_{1d} > x, Y_{1+r,d'} > x)}{P(Y_{1d} > x)}\\
&= \lim_{x\to\frac{-1}{a^*\xi}} \frac{1-\sum\limits_{h=d,d'}\exp\Big\{-\sum\limits_l\sum\limits_k(1+\xi a_{l,k,h}x)^{-1/\xi}\Big\}+\exp\Big\{-\sum\limits_{l=1}^{\infty}\sum\limits_{k=-\infty}^{\infty}[1+\xi\min(a_{l,k,d},\ a_{l,k+r-1,d'})x]^{-1/\xi}\Big\}}{1-\exp\Big\{-\sum_l\sum_k(1+\xi a_{l,k,d}x)^{-1/\xi}\Big\}}\\
&= \lim_{x\to\frac{-1}{a^*\xi}} \Big[\frac{1-2\exp\Big\{-(1+\xi a^*x)^{-1/\xi}\sum\limits_l\sum\limits_k \mathbf{1}_{\{a_{l,k,d}=a^*\}}\Big\}+\exp\Big\{-(1+\xi a^*x)^{-1/\xi}\sum\limits_l\sum\limits_k \mathbf{1}_{\{\min(a_{l,k,d},\ a_{l,k+r-1,d'})=a^*\}}}{1-\exp\Big\{-(1+\xi a^*x)^{-1/\xi}\sum\limits_l\sum\limits_k \mathbf{1}_{\{a_{l,k,d}=a^*\}}\Big\}}\\
&= \frac{2\sum\limits_l\sum\limits_k \mathbf{1}_{\{a_{l,k,d}=a^*\}} - \sum\limits_l\sum\limits_k \mathbf{1}_{\{\min(a_{l,k,d},\ a_{l,k+r-1,d'})=a^*\}}}{\sum\limits_l\sum\limits_k \mathbf{1}_{\{a_{l,k,d}=a^*\}}},
\end{aligned}$$

$\square$

Proof of **Result** 8. From (6.6) and $y = n^{\xi}x + n^{\xi}/(a^*\xi) - 1/(a^*\xi)$, $x < -1/(a^*\xi)$, we have

$$\begin{aligned}
&\mathrm{P}(\tilde{Y}_{id} < n^{\xi}x + \frac{n^{\xi}}{a^*\xi} - \frac{1}{a^*\xi},\ 1 \le i \le n) \\
&= \exp\Big\{ - n[1 + \xi a^*(n^{\xi}x + \frac{n^{\xi}}{a^*\xi} - \frac{1}{a^*\xi})]^{-1/\xi} \sum_l \sum_k \mathbf{1}_{\{a_{l,k,d}=a^*\}} \Big\} \\
&= e^{-(1+a^*\xi x)^{-1/\xi} \sum_l \sum_k \mathbf{1}_{\{a_{l,k,d}=a^*\}}}
\end{aligned}$$

which is in the domain of Weibull type.

$$\begin{aligned}
\mathrm{P}(Y_{id} <& n^{\xi}x + \frac{n^{\xi}}{a^*\xi} - \frac{1}{a^*\xi},\ 1 \le i \le n) \\
&= \exp\Big(- [1 + \xi a^*(n^{\xi}x + \frac{n^{\xi}}{a^*\xi} - \frac{1}{a^*\xi})]^{-1/\xi} \sum_l \sum_m \mathbf{1}_{\{\min\limits_{1-m\le k \le n-m} a_{l,k,d}=a^*\}} \Big) \\
&\to e^{-(1+a^*\xi x)^{-1/\xi} \sum_l \mathbf{1}_{\{\min_k a_{l,k,d}=a^*\}}}
\end{aligned}$$

which gives the extremal index for the dth sequence as follows:

$$\theta_d = \frac{\sum_l \mathbf{1}_{[\min\limits_{-\infty<k<\infty} a_{l,k,d}=a^*]}}{\sum_l \sum_k \mathbf{1}_{(a_{l,k,d}=a^*)}}.$$

From (6.6) and $y_d = n^{\xi}x_d + n^{\xi}/(a^*\xi) - 1/(a^*\xi)$, $x_d < -1/(a^*\xi)$, we have

$$\begin{aligned}
&\mathrm{P}(\tilde{Y}_{id} < n^{\xi}x_d + \frac{n^{\xi}}{a^*\xi} - \frac{1}{a^*\xi},\ 1 \le i \le n,\ d = 1, \ldots, D) \\
&= \exp\Big\{ - n \sum_l \sum_k [1 + \xi \min_d a_{l,k,d}(n^{\xi}x_d + \frac{n^{\xi}}{a^*\xi} - \frac{1}{a^*\xi})]^{-1/\xi} \mathbf{1}_{\{\min_d a_{l,k,d}=a^*\}} \Big\} \\
&= e^{-\sum_l \sum_k (1+\xi \min_d a_{l,k,d} x_d)^{-1/\xi} \mathbf{1}_{\{\min_d a_{l,k,d}=a^*\}}}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{P}(Y_{id} <& n^{\xi}x_d + \frac{n^{\xi}}{a^*\xi} - \frac{1}{a^*\xi},\ 1 \le i \le n,\ d = 1, \ldots, D) \\
&= \exp\Big(- \sum_l \sum_m [1 + \xi \min_{1-m\le k\le n-m} \min_d a_{l,k,d}(n^{\xi}x_d + \frac{n^{\xi}}{a^*\xi} - \frac{1}{a^*\xi})]^{-1/\xi} \mathbf{1}_{\{\min\limits_{1-m\le k\le n-m} \min_d a_{l,k,d}=a^*\}} \Big) \\
&\to e^{-\sum_l (1+\xi \min_k \min_d a_{l,k,d} x_d)^{-1/\xi} \mathbf{1}_{\{\min_k \min_d a_{l,k,d}=a^*\}}}.
\end{aligned}$$

Similar to the proof of Result 4, the extremal index for the multivariate processes is proved to be (2.15). $\square$

Proof of **Result** 9. Let $\delta = \sum_l \sum_k \max(a^{-1}_{l,k,d},\ a^{-1}_{l,k+r-1,d'})$, we have,

$$
\begin{aligned}
\lambda_{dd'_r} &= \lim_{x\to\infty} \frac{P(Y_{1d} > x, Y_{1+r,d'} > x)}{P(Y_{1d} > x)} \\
&= \lim_{x\to\infty} \frac{1 - 2\exp\{-\frac{1}{x^\alpha} - \frac{1}{x}\} + \exp\{-\frac{2}{x^\alpha} - \frac{\delta}{x}\}}{1 - \exp\{-\frac{1}{x^\alpha} - \frac{1}{x}\}} \\
&= \lim_{x\to\infty} \frac{2(\frac{\alpha}{x^{\alpha+1}} + \frac{1}{x^2})\exp\{-\frac{1}{x^\alpha} - \frac{1}{x}\} - (\frac{2\alpha}{x^{\alpha+1}} + \frac{\delta}{x^2})\exp\{-\frac{2}{x^\alpha} - \frac{\delta}{x}\}}{(\frac{\alpha}{x^{\alpha+1}} + \frac{1}{x^2})\exp\{-\frac{1}{x^\alpha} - \frac{1}{x}\}} \\
&\to \begin{cases} 0, & \text{if } \alpha < 1; \\ 2-\delta, & \text{if } \alpha \geq 1. \end{cases}
\end{aligned}
$$

$\square$

Proof of **Result** 10. When $\alpha < 1$, for sufficiently large x, we have

$$
\frac{1 - 2e^{-\frac{1}{x^\alpha} - \frac{1}{x}} + e^{-\frac{2}{x^\alpha} - \frac{\delta}{x}}}{(1 - e^{-\frac{1}{x^\alpha} - \frac{1}{x}})^{1/\eta}} \simeq \frac{\frac{2-\delta}{x} + \frac{1}{x^{2\alpha}} - \frac{1-2\delta}{x^{1+\alpha}} - \frac{1-\delta^2/2}{x^2}}{[\frac{1}{x^\alpha} + \frac{1}{x} - \frac{1}{2}(\frac{1}{x^{2\alpha}} + \frac{2}{x^{1+\alpha}} + \frac{1}{x^2})]^{1/\eta}}
$$

which implies (3.7). $\square$

Proof of **Result** 11. For any $a > 1$, as $y \to \infty$, we have the following properties:

$$
\begin{cases}
2y^\alpha < ay,\ (y > 1),\ e^{-y^\alpha} > e^{-ay},\ \frac{e^{-ay}}{y^{\alpha-1}e^{-y^\alpha}} = \frac{y^{1-\alpha}}{e^{(ay-y^\alpha)}} < \frac{y^{1-\alpha}}{e^{y^\alpha}} \to 0 & \text{if } \alpha < 1 \\
\frac{e^{-ay}}{e^{-y}} = e^{-(a-1)y} \to 0, & \text{if } \alpha = 1 \\
y^\alpha > ay + y,\ (y \text{ sufficiently large }),\ e^{-y^\alpha} < e^{-ay},\ \frac{y^{\alpha-1}e^{-y^\alpha}}{e^{-ay}} = \frac{y^{\alpha-1}}{e^{(y^\alpha - ay)}} < \frac{y^{\alpha-1}}{e^y} \to 0, & \text{if } \alpha > 1.
\end{cases} \quad (6.7)
$$

We have

$$
\begin{aligned}
\lambda_{dd'_r} &= \lim_{y\to\infty} \frac{P(Y_{1d} > y, Y_{1+r,d'} > y)}{P(Y_{1d} > y)} \\
&= \lim_{y\to\infty} \frac{1 - \sum\limits_{h=d,d'}(1-e^{-y^\alpha})\prod_l\prod_k(1-e^{-a_{l,k,h}y}) + (1-e^{-y^\alpha})^2\prod_{l=1}^\infty\prod_{k=-\infty}^\infty(1-e^{-\min(a_{l,k,d},\ a_{l,k+r-1,d'})y})}{1-(1-e^{-y^\alpha})\prod_l\prod_k(1-e^{-a_{l,k,d}y})} \\
&\to \begin{cases} 0, & \text{if } \alpha \leq 1; \\ \frac{2n^* - \sum_{l,k}\mathbf{1}_{\min(a_{l,k,1},a_{l,k,2})=a^*}}{n^*}, & \text{if } \alpha > 1. \end{cases}
\end{aligned}
$$

$\square$

Proof of **Result** 12. For $\alpha \leq 1$, using (6.7), we have

$$
\begin{aligned}
&\lim_{y\to\infty} \frac{1 - \sum_{s=d,d'}(1-e^{-y})\prod_l\prod_k(1-e^{-a_{lks}y}) + (1-e^{-y})^2\prod_{l=1}^\infty\prod_{k=-\infty}^\infty(1-e^{-\min(a_{l,k,d},\ a_{l,k+r,d'})y})}{[1-(1-e^{-y})\prod_l\prod_k(1-e^{-a_{l,k,d}y})]^2} \\
&\quad = 1
\end{aligned}
$$

which shows that $\eta_{dd'_r} = 1/2$.

For $\alpha > 1$ and no pairs of $(a_{l,k,1},\ a_{l,2-m,1})$, it is clear that $\eta_{dd'_r} = a^*/b_1$ using (3.13). $\square$

Proof of Lemma 13. Suppose (3.15) is true for a sequence of u_n. Then we have $ne^{-\beta u_n^\alpha} \to e^{-x}$ or $\log(n) - \beta u_n^\alpha \to -x$, which leads to

$$u_n^\alpha = \frac{1}{\beta}\{x + \log(n) + o(1)\} = \frac{\log(n)}{\beta}\{1 + \frac{x}{\log(n)} + o(\frac{1}{\log(n)})\},$$

so

$$u_n = \frac{[\log(n)]^{1/\alpha}}{\beta^{1/\alpha}}\{1 + \frac{x}{\log(n)} + o(\frac{1}{\log(n)})\},$$

which can be expressed as

$$u_n = a_n x + b_n + o(a_n),$$

where $a_n = \beta^{-1/\alpha}[\log(n)]^{1/\alpha - 1}$ and $b_n = \beta^{-1/\alpha}[\log(n)]^{1/\alpha}$. □

References

[1] Draisma, G., Drees, H., Ferreira, A., and de Haan, L. (2004). Bivariate tail estimation: dependence in asymptotic independence. *Bernoulli* 10, 251280.

[2] Embrechts, P., Klüppelberg, C. and Mikosch, T. (1997) *Modelling Extremal Events for Insurance and Finance,* Springer

[3] Embrechts, P., McNeil, A., and Straumann, D. (2002). Correlation and dependence in risk management: properties and pitfalls In: *Risk Management: Value at Risk and Beyond*, ed. M.A.H. Dempster, Cambridge University Press, Cambridge, pp. 176-223.

[4] Ferro, C. A. T. and Segers, J. (2003). Inference for clusters of extreme values. *J. Roy. Statist. Soc., B*, **65**, 545–556.

[5] de Haan, L. and Resnick, S. I. (1977). Limit theory for multivariate sample extremes. *Z. Wahrscheinlichkeitstheorie und verw. Gebiete,* **40**, 317-337.

[6] Heffernan, J. (2000). A directory of coefficients of tail dependence. *Extremes*, **3:3**, 279-290.

[7] Heffernan, J. and Tawn, J. (2004). A conditional approach for multivariate extreme values. *J. Roy. Statist. Soc. B*, **66**, pp. 497-530. (with discussion).

[8] Hsing, T. (1989) Extreme value theory for multivariate stationary sequesnces. *Journal of Multivariate Analysis.* **29**, 274-291.

[9] Hsing, T. (1993) Extremal index estimation for a weakly dependent stationary sequence. *Ann. Statist.* **21**, no. 4, 2043–2071.

[10] Joe, H. (1997). *Multivariate models and dependence concepts*, Chapman & Hall.

[11] Laurini, F., Tawn, J.A. (2003). New Estimators for the Extremal Index and Other Cluster Characteristics *Extremes*, **6**, no. 3, pp. 189–211.

[12] Leadbetter, M. R. (1983) Extremes and local dependence in stationary sequences. *Z. Wahrsch. Verw. Gebiete* **65**, no. 2, 291–306

[13] Leadbetter, M. R., Lindgren, G. and Rootzén, H. (1983). *Extremes and related properties of random sequences and processes*. Springer, Berlin.

[14] Leadbetter, M.R., Weissman, I., de Haan, L. and Rootzén, H. (1989) On clustering of high values in statistically stationary series. In: *Proc. 4th Int. Meet. Statistical Climatology* (ed: J. Sanson). Wellington: New Zealand Meteorological Service.

[15] Ledford, A. W. and Tawn, J. A. (1996). Statistics for near independence in multivariate extreme values. *Biometrika*, **83**, 169-187.

[16] Ledford, A. W. and Tawn, J. A. (1997). Modelling dependence within joint tail regions, *Journal of the Royal Statistical Society*, Series B, 59, 475.499.

[17] Ledford, A. W. and Tawn, J. A. (2003) Diagnostics for dependence within time series extremes, *Journal of the Royal Statistical Society*, Soc. B. 65, 521543

[18] Loynes, R. M. (1965) Extreme values in uniformly mixing staionary stochastic processes. *Ann. Math. Statist.* **36**, 993-999

[19] Mari, D. D. and Kotz, S. (2001) *Correlation and Dependence*. Imperial College Press.

[20] Martins, A. P. and Ferreira, H. (2005). The Multivariate Extremal Index and the Dependence Structure of a multivariate extreme value distribution. *Test*, **14**. pp433-448.

[21] Nandagopalan, S. (1990) *Multivariate extremes and the estimation of the extremal index.* Ph.D. dissertation, Dept. Of Statistics, University of North Carolina, Chapel Hill.

[22] Nandagopalan, S. (1994) On the multivariate extremal index. *J. of Research, National Inst. of Standards and Technology*. **99**, 543-550.

[23] Newell, G. F. (1964) Asymptotic extremes for m-dependent random variables. *Ann. Math. Statist.* **35**, 1322-1325

[24] O'Brien, G. L. (1974) Limit theorems for the maximum term of a stationary process. *Ann. Probab.* **2**. 540-545

[25] Peng, L. (1999). Estimation of the coefficient of tail depdendence in bivariate extremes. *Statis. Probab. Letters*. **43**, 399-409.

[26] Pickands, J. III (1981). Multivariate extreme value distributions. *Proc. 43rd Session I.S.I.* (Buenos Aires), 859-878.

[27] Resnick, S. I. (1987) *Extreme Values, Regular Variation, and Point Processes.* Springer, New York.

[28] Sibuya, M. (1960). Bivariate extreme statistics, I. *Ann. Inst. Statist. Math*, **11**, 195-210.

[29] Smith, R. L., and Weissman, I. (1996). Characterization and estimation of the multivariate extremal index. Manuscript, UNC.

[30] Weissman, I. (1994) On the extremal index of stationary sequences. Technical report, presented at the Conference on Multivariate Extreme Value Estimation with Applications to Economics and Finance, Erasmus University, Rotterdam.

[31] Zhang, Z., (2002). *Multivariate Extremes, Max-Stable Process Estimation and Dynamic Financial Modeling*, Doctoral thesis, Department of Statistics, University of North Carolina.

[32] Zhang, Z. (2003a). Quotient correlation: a sample based alternative to Pearson's correlation. Submitted for publication.

[33] Zhang, Z. (2004). On approximating max-stable processes and constructing extremal copula functions. Submitted for publication.

[34] Zhang, Z. and Huang, J. (2006) "Extremal fiancial risk model and portfolio evaluation." *Computational Statistics and Data Analysis*, **51**, 2313-2338.

[35] Zhang, Z. and Smith, R. L. (2004a). The behavior of multivariate maxima of moving maxima processes. *J. Appl. Probab.* 41.4

[36] Zhang, Z. and Smith, R.L. (2004b). "On the estimation and application of Max-stable processes." Submitted. Under revision.