

Chapter 8

Two Dichotomous Responses

8.1 Study Suggestions

Chapter 8 contains many interesting and important ideas, yet I often fail to cover it in my class. Or if I do cover Chapter 8, I only cover its first two sections. There are two explanations for this apparent paradox. First, the notation of Chapter 8 is very difficult; so cumbersome, in fact, that I wonder whether the value of the material outweighs the discomfort of learning it. Second, Chapter 8 is very different from the rest of the text. Chapter 8 is not primarily concerned with designing experiments or analyzing data. Rather, it is mostly concerned with the structure of probability on two features. Chapter 8 does, however, contain a new confidence interval formula in Section 8.5 and some important comments on sampling in Section 8.3.

In Section 8.2, I use the application of a screening test for a medical condition to motivate the definitions conditional probabilities and independence, and related formulas. Remember that these ideas are applicable to a wide range of applications beyond screening tests. I consider, however, screening tests to be the application with the greatest general interest.

Regarding inference, the main topic of Chapter 8 is the importance of noting how the sample is selected. There are three methods of random sampling that could occur in the problems of Chapter 8.

- **Sampling Method P :** A random sample is selected from the population box. (P is for population.)
- **Sampling Method A :** The population box is divided into two boxes. Box A consists of all subjects with characteristic A and box A^c con-

sists of all subjects without characteristic A . Independent random samples are selected from boxes A and A^c .

- **Sampling Method B :** The population box is divided into two boxes. Box B consists of all subjects with characteristic B and box B^c consists of all subjects without characteristic B . Independent random samples are selected from boxes B and B^c .

With sampling method P :

1. Inference may be performed validly for any individual unconditional probability, such as $P(A)$, $P(AB)$, and $P(B^c)$, using the methods of Chapter 6.
2. Inference may be performed validly for $p_{B|A} - p_{B|A^c}$ using the methods of Chapter 7.
3. Inference may be performed validly for $p_{A|B} - p_{A|B^c}$ using the methods of Chapter 7.

With sampling method A , however, only the second item in the above list is true. Finally, with sampling method B only the third item in the above list is true. Thus, it is critical that one take care to ascertain how a sample has been selected.

For sampling method P , Section 8.5 presents inference for $p_A - p_B$. Hypothesis testing for this difference reduces to the familiar McNemar's test. Confidence interval estimation of this difference, however, involves the use of a new formula. Note that for many applications, the difference $p_A - p_B$ is not of interest.

8.2 Solutions to Odd-Numbered Exercises

Solutions for Sections 8.3 and 8.4

1. (c) Overall, the point estimate of the proportion of licensed drivers who do not drink alcohol is 0.204. The point estimate of the proportion of female licensed drivers who do not drink alcohol is 0.232. The point estimate of the proportion of male licensed drivers who do not drink alcohol is 0.178. There is a 5.4 percentage point difference between the female and male point estimates.
- (d) Overall, the point estimate of the proportion of licensed drivers who are female is 0.487. Of the drivers who do not drink alcohol, the point estimate of the proportion who are female is 0.552. Of the drivers who do drink alcohol, the point estimate of the proportion who are female is 0.470. There is an 8.2-percentage-point difference between the estimates of the proportions of female drivers.
- (e) The standardized value of the test statistic for Fisher's test,

$$z = \frac{\sqrt{2,339}[264(986) - 876(214)]}{\sqrt{1,140(1,200)(478)(1,862)}}$$
 is 3.19. The approximate P-value for the third alternative is twice the area to the right of 3.19 under the standard normal curve. This area equals $2(0.0007) = 0.0014$. Thus, I reject the null hypothesis that the variables are independent.
- (g) I conclude that the proportion of female drivers who do not drink is larger than the proportion of male drivers who do not drink. At the 95 percent confidence level, I conclude that the difference between these proportions is at least 2.1 and at most 8.7 percentage points.
3. (c) The standardized value of the test statistic for Fisher's test,

$$z = \frac{\sqrt{2,126}[476(623) - 560(468)]}{\sqrt{1,036(1,091)(944)(1,183)}}$$

is 1.41. The approximate P-value for the third alternative is twice the area to the right of 1.41 under the standard normal curve. This area equals $2(0.0793) = 0.1586$. Thus, I do not reject the null hypothesis that the variables are independent.

- (a) The 95 percent confidence interval is

$$(0.460 - 0.429) \pm 1.96 \sqrt{\frac{476(560)}{(1,036)^3} + \frac{468(623)}{(1,091)^3}} = 0.031 \pm 0.042 = [-0.011, 0.073].$$

5. (c) The standardized value of the test statistic for Fisher's test,

$$z = \frac{\sqrt{667}[17(330) - 103(218)]}{\sqrt{120(548)(235)(433)}}$$

is -5.32 . The approximate P-value for the second alternative is the area to the right of 5.32 under the standard normal curve. This area is smaller than 0.0002. Thus, I reject the null hypothesis that the variables are independent.

- (d) The 95 percent confidence interval is

$$(0.142 - 0.398) \pm 1.96 \sqrt{\frac{17(103)}{(120)^3} + \frac{218(330)}{(548)^3}} = -0.256 \pm 0.075 = [-0.331, -0.181].$$

7. (b) The standardized value of the test statistic for Fisher's test is

$$z = \frac{\sqrt{234}[4(123) - 13(95)]}{\sqrt{17(218)(99)(136)}} = -1.61.$$

The approximate P-value for the second alternative is the area to the right of 1.61 under the standard normal curve. This area equals 0.0537. Thus, I barely fail to reject the null hypothesis that the variables are independent. Even though there is a huge difference in sample proportions (as discussed in part (a) above), the difference is based on so little data that it could reasonably be attributed to chance.

Solutions for Section 8.5

3. The point estimate is $\hat{p} = 980/2,240 = 0.438$.
The 99 percent confidence interval is

$$0.438 \pm 2.576 \sqrt{\frac{0.438(0.562)}{2,240}} =$$

$$0.438 \pm 0.027 = [0.411, 0.465].$$

5. The point estimate is $\hat{p} = 1,079/2,324 = 0.464$. The 99 percent confidence interval is

$$0.4642 \pm 2.576 \sqrt{\frac{0.464(0.536)}{2,324}} =$$

$$0.464 \pm 0.027 = [0.437, 0.491].$$

7. The standardized value of the test statistic is

$$z = \frac{757 - 281}{\sqrt{757 + 281}} = 14.77.$$

The approximate P-value for the third alternative is twice the area to the right of 14.77 under the standard normal curve. This area is smaller than 0.0004.

Next,

$$\sqrt{\frac{2,171(757 + 281) - (757 - 281)^2}{2,170(2,171)^2}}$$

equals 0.01408. The 99 percent confidence interval is

$$\left(\frac{757 - 281}{2,171}\right) \pm 2.576(0.01408) =$$

$$0.219 \pm 0.036 = [0.183, 0.255].$$

9. The standardized value of the test statistic is

$$z = \frac{625 - 219}{\sqrt{625 + 219}} = 13.98.$$

The approximate P-value for the third alternative is twice the area to the right of 13.98 under the standard normal curve. This area is smaller than 0.0004.

Next,

$$\sqrt{\frac{2,265(625 + 219) - (625 - 219)^2}{2,264(2,265)^2}}$$

equals 0.01226. The 99 percent confidence interval is

$$\left(\frac{625 - 219}{2,265}\right) \pm 2.576(0.01226) =$$

$$0.179 \pm 0.032 = [0.147, 0.211].$$

11. The standardized value of the test statistic is

$$z = \frac{589 - 299}{\sqrt{589 + 299}} = 9.73.$$

The approximate P-value for the third alternative is twice the area to the right of 9.73 under the standard normal curve. This area is smaller than 0.0004.

Next,

$$\sqrt{\frac{2,261(589 + 299) - (589 - 299)^2}{2,260(2,261)^2}}$$

equals 0.01290. The 99 percent confidence interval is

$$\left(\frac{589 - 299}{2,261}\right) \pm 2.576(0.01290) =$$

$$0.128 \pm 0.033 = [0.095, 0.161].$$

13. The standardized value of the test statistic is

$$z = \frac{508 - 526}{\sqrt{508 + 526}} = -0.56.$$

The approximate P-value for the third alternative is twice the area to the right of 0.56 under the standard normal curve. This area equals $2(0.2877) = 0.5754$.

Next,

$$\sqrt{\frac{2,141(508 + 526) - (508 - 526)^2}{2,140(2,141)^2}}$$

equals 0.01502. The 99 percent confidence interval is

$$\left(\frac{508 - 526}{2,141}\right) \pm 2.576(0.01502) =$$

$$-0.008 \pm 0.039 = [-0.047, 0.031].$$

15. The standardized value of the test statistic is

$$z = \frac{324 - 381}{\sqrt{324 + 381}} = -2.15.$$

The approximate P-value for the third alternative is twice the area to the right of 2.15 under the standard normal curve. This area equals $2(0.0158) = 0.0316$.

Next,

$$\sqrt{\frac{2,237(324 + 381) - (324 - 381)^2}{2,236(2,237)^2}}$$

equals 0.01186. The 99 percent confidence interval is

$$\left(\frac{324 - 381}{2,237}\right) \pm 2.576(0.01186) = -0.025 \pm 0.031 = [-0.056, 0.006].$$

8.3 Exam Questions

Questions 1–11 refer to the following data. A random sample of 100 high school seniors are asked to locate France and Japan on a map of the world. The results are summarized in the following table:

	Japan		
France	B	B^c	Total
A	44	26	70
A^c	16	14	30
Total	60	40	100

The event A means the subject could correctly locate France, and B means the subject could correctly locate Japan.

- What proportion of the sample could correctly locate France?
- What proportion of the sample could correctly locate neither France nor Japan?
- Which of the following describes $P(B|A)$?
 - It is the proportion of the population that could locate Japan and France.
 - It is the proportion of those in the population who could locate Japan who could also locate France.
 - It is the proportion of those in the population who could locate France who could also locate Japan.
 - It is the proportion of the population who could locate Japan.
 - It is the proportion of the population who could locate France.
- What is the point estimate of $P(B|A)$?
- What is the point estimate of the proportion of the population who could locate France minus the proportion of the population who could locate Japan?
- A researcher wants a 95 percent confidence interval for the proportion of the population who could correctly locate Japan. **DO NOT PERFORM ANY COMPUTATIONS**, simply replace all of the symbols in each formula by the appropriate numbers.
- A researcher wants a 95 percent confidence interval for the difference between the proportion of the population who could correctly locate France and the proportion of the population who could correctly locate Japan. **DO NOT PERFORM ANY COMPUTATIONS**, simply replace all of the symbols in each formula by the appropriate numbers.
- A researcher wants to use a 95 percent confidence interval to investigate whether the ability to locate France depends on the ability (or inability) to locate Japan. **DO NOT PERFORM ANY COMPUTATIONS**, simply replace all of the symbols in each formula by the appropriate numbers.
- A researcher wants to use the standard normal curve to obtain an approximate P-value for testing the null hypothesis that the proportion of the population who could correctly locate both Japan and France equals 0.50. Write down the

test statistic. DO NOT PERFORM ANY COMPUTATIONS, simply replace all of the symbols in each formula by the appropriate numbers.

10. A researcher wants to use the standard normal curve to obtain an approximate P-value for testing the null hypothesis that the proportion of the population who could correctly locate France equals the proportion of the population who could correctly locate Japan. Write down the test statistic. DO NOT PERFORM ANY COMPUTATIONS, simply replace all of the symbols in each formula by the appropriate numbers.
11. A researcher plans to take a second random sample of size 200 from the population. What is the 95 percent prediction interval for the number of subjects in the the second sample who will be able to locate France? DO NOT PERFORM ANY COMPUTATIONS, simply replace all of the symbols in each formula by the appropriate numbers.
12. Consider the screening test for HIV used by the Red Cross to make the blood supply safer. An official states

Given a unit of infected blood, there is a very high chance that our test will detect the infection.

In other words, the official is saying that one of the conditional probabilities is large. Which one is it? (Your answer should be of the form $p_{C|D}$, where C and D are chosen from A , A^c , B , and B^c . Recall that A means the blood is infected and B means the screening test is positive.)

13. Consider the screening test for HIV used by the Red Cross to make the blood supply safer. An official states:

Given a unit of uninfected blood, there is a very small chance that our test will incorrectly discard it.

In other words, the official is saying that one of the conditional probabilities is small; which one? (The answer should be of the form $p_{C|D}$, where C and D are chosen from A , A^c , B , and

B^c . Recall that A means the blood is infected and B means the screening test is positive.)

14. Consider the screening test for HIV used by the Red Cross to make the blood supply safer. An official states

Given a unit of uninfected blood, there is a very high chance that our test will correctly say it is safe.

In other words, the official is saying that one of the conditional probabilities is large. Which one is it? (Your answer should be of the form $p_{C|D}$, where C and D are chosen from A , A^c , B , and B^c . Recall that A means the blood is infected and B means the screening test is positive.)

15. Consider the screening test for HIV used by the Red Cross to make the blood supply safer. An official states

Given a unit of infected blood, there is a very small chance that our test will incorrectly say it is safe.

In other words, the official is saying that one of the conditional probabilities is small. Which one is it? (Your answer should be of the form $p_{C|D}$, where C and D are chosen from A , A^c , B , and B^c . Recall that A means the blood is infected and B means the screening test is positive.)

16. In the movie *Joe Versus the Volcano* the character played by Tom Hanks is diagnosed as having a “brain cloud” when in fact he does not. This is an example of a _____ test result.

- (a) correct positive
- (b) correct negative
- (c) false positive
- (d) false negative

17. A population consists of 100 persons, each of whom is asked to locate Spain and Chile on a map of the world. The results are summarized in the following table:

Spain	Chile		Total
	B	B^c	
A	50	25	75
A^c	15	10	25
Total	65	35	100

The event A means the subject could locate Spain, and B means the subject could locate Chile.

- (a) What proportion of the population could locate Spain?
- (b) What proportion of the population could locate Chile, but not Spain?
- (c) Of those in the population who could locate Spain, what proportion could not locate Chile?
- (d) What proportion of the population could locate both Spain and Chile?
- (e) Which of the following describes $p_{A^c|B}$?
- It is the proportion of the population that could locate Chile, but not Spain.
 - Of those in the population who could locate Chile, it is the proportion who could not locate Spain.
 - Of those in the population who could not locate Spain, it is the proportion who could locate Chile.
 - It is the proportion of the population who could not locate Spain.
 - It is the proportion of the population who could locate Chile.
- (f) Compute $p_{B|A^c}$.
18. A population consists of 10,000 persons, each of whom has a condition (A) or not (A^c). In addition, if given a screening test, a person would yield a positive (B) or negative (B^c) result. The population is summarized in the following table:

	B	B^c	Total
A	95	5	100
A^c	99	9,801	9,900
Total	194	9,806	10,000

- (a) What proportion of the population has the condition?
- (b) Of those in the population who have the condition, what proportion would obtain a correct screening test result?
- (c) What proportion of the population would obtain an incorrect screening test result (that is, either a false positive or a false negative)?
- (d) Of those in the population who test positive, what proportion would be condition free?

19. In the movie *Short Time* a policeman and a bus driver are tested for the same condition. The bus driver has the condition, and the policeman does not. The test results are inadvertently reversed, and the policeman is told he has the condition and the bus driver is told that he does not. The report actually given to the bus driver is an example of a _____ test result.

- (a) Correct positive
- (b) Correct negative
- (c) False positive
- (d) False negative

20. The table of population proportions for a condition and a screening test is given below.

Condition	Screening test		Total
	Positive	Negative	
Present	0.009	0.001	0.010
Absent	0.006	0.984	0.990
Total	0.015	0.985	1.000

Determine the numbers described below.

- (a) What proportion of the population has the condition?
- (b) What proportion of the population has the condition and would test negative?
- (c) What is the probability of a positive test given that the condition is present?
- (d) Of those who test positive, what proportion does not have the condition?

21. I plan to shoot one free throw. After the shot my canine companion Casey will either bark or not. You are given the following information.

- The probability that I make the shot equals 0.70.
- Given that I make the shot, the probability is 0.80 that Casey will bark.
- Given that I miss the shot, the probability is 0.60 that Casey will not bark.

- (a) Compute the probability that Casey will bark.
- (b) Given that Casey barks, what is the probability that I made the shot?

22. My near-sighted neighbor, Reg Wexford, plans to shoot one free throw. After Reg's shot, his dog will either bark or not. You are given the following information.

- The probability that Reg makes the shot equals 0.60.
- Given that Reg makes the shot, the probability is 0.80 that his dog will bark.
- Given that Reg misses the shot, the probability is 0.40 that his dog will bark.

- (a) Compute the probability that Reg's dog will bark.
- (b) Given that Reg's dog barks, what is the probability that he made the shot?

8.4 Solutions to Exam Questions

1. 0.70.

2. 0.14.

3. (c).

4. $44/70 = 0.629$.

5. 0.10.

6. The confidence interval is

$$0.6 \pm 1.96\sqrt{\frac{0.6(0.4)}{100}}$$

7. The confidence interval is

$$0.1 \pm 1.96\sqrt{\frac{100(26 + 16) - (26 - 16)^2}{99(100)^2}}$$

8. The confidence interval is

$$\left(\frac{44}{60} - \frac{26}{40}\right) \pm 1.96\sqrt{\frac{44(16)}{60^3} + \frac{26(14)}{40^3}}$$

9. The test statistic is

$$z = \frac{44 - 100(0.5)}{\sqrt{100(0.5)(0.5)}} \text{ or } z = \frac{88 - 100}{\sqrt{100}}$$

10. The test statistic is

$$z = \frac{26 - 16}{\sqrt{26 + 16}}$$

11. The prediction interval is

$$200(0.70) \pm 1.96\sqrt{200(0.7)(0.3)}\sqrt{1 + \frac{200}{100}}$$

12. $p_{B|A}$.

13. $p_{B|A^c}$.

14. $p_{B^c|A^c}$.

15. $p_{B^c|A}$.

16. (c) false positive.

17. (a) 0.75.

(b) 0.15.

(c) $25/75 = 0.33$.

(d) 0.50.

(e) ii.

(f) $15/25 = 0.60$.

18. (a) $p_A = 100/10,000 = 0.01$

(b) $p_{B|A} = 95/100 = 0.95$

(c) $p_{A^cB} + p_{AB^c} = (99+5)/10,000 = 0.0104$.

(d) $p_{A^c|B} = 99/194 = 0.5103$

19. (d).

20. (a) 0.010.

- (b) 0.001.
 (c) $0.009/0.010 = 0.90$.
 (d) $0.006/0.015 = 0.40$.

21. Let A be the event that the shot is made and let B be the event that Casey barks. The following probabilities are given.

$$P(A) = 0.7, P(B|A) = 0.8, \text{ and}$$

$$P(B^c|A^c) = 0.6.$$

Thus, $P(A^c) = 0.3$,

$$P(AB) = P(A)P(B|A) = 0.7(0.8) = 0.56,$$

and $P(A^cB^c) =$

$$P(A^c)P(B^c|A^c) = 0.3(0.6) = 0.18.$$

This allows one to construct the following table.

Shot	Barks?		Total
	Yes	No	
Make	0.56	0.14	0.70
Miss	0.12	0.18	0.30
Total	0.68	0.32	1.00

- (a) 0.68.
 (b) $0.56/0.68 = 0.824$.

22. Let A be the event that the shot is made and let B be the event that the dog barks. The following probabilities are given.

$$P(A) = 0.6, P(B|A) = 0.8, \text{ and}$$

$$P(B|A^c) = 0.4.$$

Thus, $P(A^c) = 0.4$,

$$P(AB) = P(A)P(B|A) = 0.6(0.8) = 0.48,$$

and $P(A^cB) =$

$$P(A^c)P(B|A^c) = 0.4(0.4) = 0.16.$$

This allows one to construct the following table.

Shot	Barks?		Total
	Yes	No	
Make	0.48	0.12	0.60
Miss	0.16	0.24	0.40
Total	0.64	0.36	1.00

- (a) 0.64.
 (b) $0.48/0.64 = 0.75$.