

Chapter 6

Inference for a Population

6.1 Study Suggestions

Let X be a random variable with the binomial sampling distribution with parameters n and p . In Chapter 5 you learned that X can arise when taking a random sample from a finite population, or when observing a sequence of Bernoulli trials. In Chapter 5 the focus was on using the values of n and p to compute the probability of an event involving X . Chapter 6 considers the more realistic situation in which the value of p is unknown. In Chapter 6 you learn how the observed value of X can be used to draw an inference about the unknown value of p .

The first example of inference is the point estimate. The proportion of successes *in the sample*, $\hat{p} = x/n$, is the point estimate of the proportion of successes *in the population*. The main advantage of the point estimate is that it is easy to understand and to interpret, and its main disadvantage is that it is usually incorrect. The confidence interval estimate sacrifices some of the simplicity of a point estimate to achieve a high confidence of being correct. (Remember that the terms correct and incorrect do not refer to the estimate being *computed* without or with error; they refer to whether the stated estimate equals, in the case of a point estimate, or contains, in the case of an interval estimate, the actual value of p .)

A confidence interval estimate has two parts to it—the interval and the confidence level. The confidence level is selected by the researcher and the interval is computed from the data. For example, if a researcher selects 95 percent confidence, then the

interval is

$$\hat{p} \pm 1.96 \sqrt{\frac{\hat{p}\hat{q}}{n}}.$$

The “95 percent confidence” in the 95 percent confidence interval is interpreted as follows. Before collecting data, the probability is 95 percent that the data to be obtained will yield an interval that includes the actual value of p . My students find Table 6.2 and Figure 6.1 on page 197 of the text to be helpful in their achieving an understanding of what “confidence” means. In particular, note that in a real application in which the value of p is unknown, a researcher will never know whether a confidence interval is correct. If you find this uncertainty unacceptable, then you will probably never consider confidence intervals to be of much practical value. I view a confidence interval as a reasonable middle position between the extremes of taking a census (which is usually prohibitively expensive and time consuming), and collecting no data.

In the majority of applications I have seen, the researcher simply wants to estimate the value of p . Occasionally, the researcher will consider one of the possible values of p to be of special interest. In this latter case, the researcher can perform a hypothesis test with the null hypothesis stating that p equals the special value of interest, p_0 . With the solid background of Fisher’s test, my students find the hypothesis testing in Chapter 6 to be fairly easy to understand. Formula 6.11 on page 212 of the text is an alternate expression for the standardized value of the test statistic, z , that is valid when the special value of interest is 0.5. If I am not careful in my presentation, a common error among my students is to use this expression even when p_0 is not 0.5. Remember

the restricted applicability of this formula!

Recall that McNemar's test was introduced in Section 4.3 for the randomized pairs design. Example 6.17 on pages 212–13 discusses why the test of $p = 0.5$ is identical to McNemar's test. Few of my students seem interested in this identity, and I suggest this material as optional extra reading.

Section 6.5, Predictions, is optional, but I hope your teacher will decide to cover this material. Some of the reasons I like this material are listed below.

- For Bernoulli trials, making a prediction is arguably more interesting and natural than constructing a confidence interval or performing a hypothesis test.
- An analyst never knows whether a confidence interval is correct or whether the conclusion of a hypothesis test is correct. By contrast, one eventually learns whether a prediction is correct. For my students, this distinction makes prediction much more natural and satisfying than estimation and testing.
- The discussion of prediction provides the instructor with an opportunity to discuss the importance of the critical assumption of this methodology, namely that the future will be like the past—that is, that the value of p does not change, and that the assumption of Bernoulli trials remains valid. (I welcome opportunities for a meaningful discussion of the validity of assumptions.)

6.2 Solutions to Odd-Numbered Exercises

Solutions for Section 6.2

- (a) The point estimate is $\hat{p} = 552/756 = 0.730$. The 95 percent confidence interval is

$$0.730 \pm 1.96 \sqrt{\frac{0.730(0.270)}{756}} =$$

$$0.730 \pm 0.032 = [0.698, 0.762].$$
- (b) The point estimate is $\hat{p} = 340/756 = 0.450$. The 95 percent confidence interval is

$$0.450 \pm 1.96 \sqrt{\frac{0.450(0.550)}{756}} =$$

$$0.450 \pm 0.035 = [0.415, 0.485].$$
- (c) Seventy-three percent of those surveyed said that they were willing to pay higher prices for food grown with fewer pesticides and chemicals, but only 45 percent of these same people buy organic foods. Thus, it appears that for many persons interviewed, behavior does not match belief.
- The point estimate is $\hat{p} = 469/756 = 0.620$. The 90 percent confidence interval is

$$0.620 \pm 1.645 \sqrt{\frac{0.620(0.380)}{756}} =$$

$$0.620 \pm 0.029 = [0.591, 0.649].$$
- The point estimate is $\hat{p} = 22/37 = 0.595$. The 90 percent confidence interval is

$$0.595 \pm 1.645 \sqrt{\frac{0.595(0.405)}{37}} =$$

$$0.595 \pm 0.133 = [0.462, 0.728].$$
- The point estimate is $\hat{p} = 29/47 = 0.617$. The 95 percent confidence interval is

$$0.617 \pm 1.96 \sqrt{\frac{0.617(0.383)}{47}} =$$

$$0.617 \pm 0.139 = [0.478, 0.756].$$
- (a) The point estimate is $\hat{p} = 21/50 = 0.420$. The 98 percent confidence interval is

$$0.420 \pm 2.326 \sqrt{\frac{0.420(0.580)}{50}} =$$

$$0.420 \pm 0.162 = [0.258, 0.582].$$

- (b) The point estimate is $\hat{p} = 20/50 = 0.400$.
The 98 percent confidence interval is

$$0.400 \pm 2.326 \sqrt{\frac{0.400(0.600)}{50}} =$$

$$0.400 \pm 0.161 = [0.239, 0.561].$$

11. The point estimate is $\hat{p} = 400/417 = 0.959$.
The 95 percent confidence interval is

$$0.959 \pm 1.96 \sqrt{\frac{0.959(0.041)}{417}} =$$

$$0.959 \pm 0.019 = [0.940, 0.978].$$

13. (a) A positive test result at six months of age would be inconclusive; it could represent HIV infection or it could represent the presence of maternal antibodies without HIV infection.
- (b) It is difficult to say why. Perhaps many of the 308 children had not yet reached the age of 18 months. Perhaps many children were lost to follow-up because their mothers did not return to the clinic.
- (c) The point estimate is $\hat{p} = 76/117 = 0.650$. The 90 percent confidence interval is

$$0.650 \pm 1.645 \sqrt{\frac{0.650(0.350)}{117}} =$$

$$0.650 \pm 0.073 = [0.577, 0.723].$$

Solutions for Section 6.3

- As the confidence level increases, the width of the confidence interval also increases. As a result, the confidence levels are 95, 99, 90, 80, and 98 percent, respectively.
- Refer to Figure 6.3 on page 200 of the text. Victor's half-width is the smaller of the two half-widths, so his point estimate must be farther from 0.5 than is Daniel's. Thus, Victor had $\hat{p} = 0.20$ and Daniel obtained $\hat{p} = 0.40$.

Solutions for Section 6.4

- Oscar has the largest value of X , so he provides the strongest evidence in support of the first alternative.
 - Victor has the smallest value of X , so he provides the strongest evidence in support of the second alternative.
 - Note that $np_0 = 1,000(0.20) = 200$. Victor's value of X is the farthest from 200 of the three metasubjects; thus, he provides the strongest evidence in support of the third alternative.
- The standardized value of Pedro's result, $X = 90$, is

$$z = \frac{90 - 400(0.20)}{\sqrt{400(0.20)(0.80)}} = 1.25.$$

- The approximate P-value for the first alternative is the area under the standard normal curve to the right of 1.25. This area is 0.1056.
 - The approximate P-value for the third alternative is twice the area under the standard normal curve to the right of 1.25. This area is $2(0.1056) = 0.2112$.
9. The standardized value of $X = 469$ is

$$z = \frac{469 - 756(0.50)}{\sqrt{756(0.50)(0.50)}} = 6.62.$$

Alternatively, since $p_0 = 0.5$,

$$z = \frac{2(469) - 756}{\sqrt{756}} = 6.62.$$

The approximate P-value for the first alternative is the area under the standard normal curve to the right of 6.62. This area is smaller than 0.0002.

11. The standardized value of $X = 57$ is

$$z = \frac{57 - 96(0.50)}{\sqrt{96(0.50)(0.50)}} = 1.84.$$

Alternatively, since $p_0 = 0.5$,

$$z = \frac{2(57) - 96}{\sqrt{96}} = 1.84.$$

The approximate P-value for the third alternative is twice the area under the standard normal curve to the right of 1.84. This area is $2(0.0329) = 0.0658$.

13. The standardized value of $X = 184$ is

$$z = \frac{184 - 193(0.80)}{\sqrt{193(0.80)(0.20)}} = 5.33.$$

The approximate P-value for the third alternative is twice the area under the standard normal curve to the right of 5.33. This area is smaller than 0.0004.

Solutions for Section 6.5

1. The point prediction of the number of heads is $10,000(0.5) = 5,000$. The 95 percent prediction interval is

$$5,000 \pm 1.96\sqrt{10,000(0.5)(0.5)} = \\ 5,000 \pm 98 = [4,902, 5,098].$$

3. (a) Note that p is assumed to be known and equal to 0.5. The point prediction is $mp = 200(0.5) = 100$ female babies.

To obtain the 90 percent prediction interval, first compute

$$100 \pm 1.645\sqrt{200(0.5)(0.5)} = \\ 100 \pm 11.63 = [88.37, 111.63].$$

After rounding, the 90 percent prediction interval is [88, 112].

- (b) Note that the value of p is unknown. The training set of 100 births yields $\hat{p} = 52/100 = 0.52$. The point prediction is $m\hat{p} = 200(0.52) = 104$ female babies.

To obtain the 90 percent prediction interval, first compute $104 \pm$

$$1.645\sqrt{200(0.52)(0.48)}\sqrt{1 + 200/100} = \\ 104 \pm 20.13 = [83.87, 124.13].$$

After rounding, the 90 percent prediction interval is [84, 124]. Note that this interval is much wider—40 versus 24 babies—than the interval for p known.

5. First, note that the value of p is unknown. The first 48 trials on Muffin yield $\hat{p} = 27/48 = 0.562$.

(a) The point prediction is $m\hat{p} = 48(27/48) = 27$.

- (b) To obtain the 90 percent prediction interval, first compute $27 \pm$

$$1.645\sqrt{48(0.562)(0.438)}\sqrt{1 + 48/48} = \\ 27 \pm 8.00 = [19.00, 35.00].$$

There is no need to round off this answer; the 90 percent prediction interval is [19, 35].

- (c) The point prediction was three successes too low. The prediction interval was correct.

Solutions for Section 6.6

1. The probability that any particular person scores 29 or more correct is 0.0200. Thus, the probability that at least one person of five scores 29 or more correct is $1 - (0.98)^5 = 0.0961$.
3. Note that $m = 10$ and $\alpha = 0.05$.

(a) The probability that all 10 intervals will be correct is greater than or equal to $1 - \alpha m = 1 - 0.05(10) = 0.50$.

(b) Because of the assumed independence, the probability that all 10 intervals will be correct is equal to $(1 - \alpha)^{10} = 0.5987$.

(c) The second answer is much more informative.

6.3 Exam Questions

1. A confidence interval for p is called correct if _____.

- (a) it contains the true value of p
 (b) it contains the special value of interest p_0
 (c) it is sufficiently narrow to be useful
 (d) no computational errors were made in obtaining it

2. Casey's favorite brand of dog biscuits comes in seven flavors—bacon, cheese, liver, meat, milk, poultry, and vegetable. Her owner purchased a box of biscuits and sorted the biscuits by flavor. The following frequencies were obtained.

Bacon—29, cheese—26, liver—37,
meat—25, milk—34, poultry—27,
and vegetable—29.

(Hint: The total number of biscuits is 207.) Casey's owner plans to buy a new box containing 140 biscuits. In order to obtain answers to the questions below you may assume that the process of placing biscuits in the box at the factory has the following property. For liver flavor defined to be a success and any other flavor a failure, the placement of successive biscuits in a box satisfy the assumptions of Bernoulli trials. DO NOT ASSUME, however, that the seven flavors are necessarily equally likely.

- (a) Compute the point prediction of the number of liver flavored biscuits in the new box of 140 biscuits.
 - (b) Compute the 90% prediction interval for the number of liver flavored biscuits in the new box of 140 biscuits.
3. A researcher tested 100 adult cats and classified 65 as 'ambidextrous' and the remaining 35 as having one paw dominate. Assume that these cats are a random sample from a population of cats. The researcher plans to test an additional 150 cats selected at random from the population. Compute a 90% prediction interval for the number of these 150 new cats that are ambidextrous. (Remember, you cannot have fractional cats.)
4. Kramer selects a random sample of size 100 from a dichotomous population and obtains 40 successes and 60 failures. Construct the 90% confidence interval for p .
5. George obtains one set of data and constructs three confidence intervals for p . The levels of these intervals are 80%, 90%, and 95%. The intervals are $[0.658, 0.742]$, $[0.636, 0.764]$, and

$[0.647, 0.753]$. Match each interval with its level.

6. Mary enjoys shooting free throws at the outdoor basketball court in her neighborhood. In fact, every morning Mary attempts 200 free throws. One cloudy day she made 75 of her first 100 hundred shots. After 100 shots it began to rain hard and the wind velocity increased dramatically. On this day Mary finished with 100 successful shots in 200 attempts.

Nancy examines Mary's data and states, "My point estimate of the probability Mary makes a free throw is 0.50."

Do you agree with Nancy? Explain your answer.

7. You are given the following information.

- On Monday afternoon a golf pro attempted fifty putts from the distance of five feet, and obtained 32 successes (made putts) and 18 failures (missed putts).
- On Tuesday afternoon, the same pro repeated the study and obtained 34 successes and 16 failures.
- Assume that the shots on Monday were Bernoulli trials with probability of success equal to p_2 .
- Assume that the shots on Tuesday were Bernoulli trials with probability of success equal to p_1 .
- Note that p_2 and p_1 might be equal or unequal.

Construct the 90 percent confidence for p_2 .

8. Refer to the previous question. Imagine that it is Monday evening and you want to use Monday afternoon's data to predict how the golf pro would perform on Tuesday afternoon.

- (a) Assuming that $p_1 = p_2$, construct the 80 percent prediction interval for the number of successes the pro would obtain on Tuesday afternoon.

- (b) Does your prediction interval turn out to be correct? Explain your answer.
- (c) Briefly explain why it is necessary to assume that $p_1 = p_2$.
9. The table below displays probabilities for the Bin(20,0.4) distribution.

x	$P(X = x)$	$P(X \leq x)$	$P(X \geq x)$
1	0.0005	0.0005	1.0000
2	0.0031	0.0036	0.9995
3	0.0124	0.0160	0.9964
4	0.0350	0.0510	0.9840
5	0.0746	0.1256	0.9490
6	0.1244	0.2500	0.8744
7	0.1659	0.4159	0.7500
8	0.1797	0.5956	0.5841
9	0.1597	0.7553	0.4044
10	0.1171	0.8725	0.2447
11	0.0710	0.9435	0.1275
12	0.0355	0.9790	0.0565
13	0.0146	0.9935	0.0210
14	0.0049	0.9984	0.0065
15	0.0013	0.9997	0.0016
16	0.0003	1.0000	0.0003

Find the exact P-value for testing $H_0 : p = 0.4$ versus $H_1 : p > 0.4$ if $n = 20$ and the observed value of X is 13.

10. Mieke performed 40 trials. Each trial was playing a concert B \flat on her clarinet into a tuner. The tuner then reports whether the note is sharp, flat, or perfectly in tune. Define *perfectly in tune* a success, and either *sharp* or *flat* a failure. Mieke obtained a total of 26 successes and 14 failures. Assume that the 40 trials satisfy the assumption of Bernoulli trials. Mieke plans to perform 60 more trials and wants to predict the total number of successes that she will obtain in those 60 trials. Assume that the 60 future trials are Bernoulli trials and assume that Mieke's ability (her probability of success) is the same for the 60 future trials as it was for the 40 previous trials.
- (a) Compute the point prediction of the total number of successes Mieke will obtain in the 60 future trials.

- (b) Construct the 90 percent prediction interval for the total number of successes Mieke will obtain in the 60 future trials.

- (c) After all of these computations, Mieke performs the additional trials and obtains 47 successes and 13 failures.

Was the point prediction correct?

Was the 90 percent prediction interval correct?

11. Rachel and Sam select random samples from the same population. Rachel's sample size is n , and Sam's sample size is $4n$. Rachel computes two confidence intervals for p —90 percent and 95 percent—from her data, and Sam computes one confidence interval for p —95 percent—from his data. Here are the intervals:

$$[0.39, 0.49], [0.31, 0.51], [0.33, 0.49].$$

Match each interval above with its researcher and level.

12. Find the exact P-value for testing $H_0 : p = 0.5$ versus $H_1 : p < 0.5$ if $n = 12$ and the observed value of X is 3.
13. Bob is an avid bridge player. It can be shown that if the cards are shuffled thoroughly, the probability that a bridge hand contains no aces is 0.3038. For the next 1,000 hands of bridge Bob is dealt, find the 90 percent prediction interval for the number that contain no aces.
14. Michael enjoys shooting free throws. Assume that his attempts satisfy the assumptions of Bernoulli trials, but all we know about Michael's ability is that he has attempted 500 free throws and obtained 350 successes. True or false? Michael's probability of success equals 0.700.
15. A random sample of size 500 yields 120 successes. Use these data to test $H_0 : p = 0.20$ versus $H_1 : p \neq 0.20$.

6.4 Solutions to Exam Questions

1. (a).
2. (a) This is an example of prediction with p unknown. Note that $x = 37$, $n = 207$, and $m = 140$. First, compute $140(37/207) = 25.02$. Thus, the point prediction is 25.
- (b) Note that $37/207 = 0.179$. Thus,

$$\sqrt{m\hat{p}\hat{q}} = \sqrt{140(0.179)(0.821)} = 4.536,$$

$$\sqrt{1 + m/n} = \sqrt{1 + 140/207} = 1.295.$$

$$25.02 \pm 1.645(4.536)(1.295) =$$

$$25.02 \pm 9.66 = [15.36, 34.68].$$

After rounding, the 90% prediction interval for the number of liver flavored biscuits is [15, 35].

3. $\sqrt{150(0.65)(0.35)} = 5.842$.
 $\sqrt{1 + 150/100} = 1.581$. Thus, the prediction interval is
 $150(0.65) \pm 1.645(5.842)(1.581) =$
 $97.5 \pm 15.2 = [82.3, 112.7]$.
 After rounding, the 90 percent prediction interval is [82, 113].
- 4.

$$0.40 \pm 1.645\sqrt{\frac{0.4(0.6)}{100}} =$$

$$0.40 \pm 0.08 = [0.32, 0.48].$$

5. The 80 percent confidence interval is the narrowest and the 95 percent confidence interval is the widest of the three. Thus, [0.658, 0.742] is 80 percent; [0.636, 0.764] is 95 percent; and [0.647, 0.753] is 90 percent.
6. A point estimate should be computed only if you are willing to assume that the data are a random sample from a population. For subjects that are trials, this means you must be willing to

assume the trials are Bernoulli trials. This assumption does not seem reasonable for the situation described in the problem. In particular, the second assumption of Bernoulli trials appears to be invalid.

Thus, I disagree with Nancy.

7. The 90 percent confidence for p_2 is

$$0.64 \pm 1.645\sqrt{\frac{0.64(0.36)}{50}} =$$

$$0.64 \pm 0.11 = [0.53, 0.75].$$

8. (a) Monday's data provide an estimate of the unknown value of p ; namely, $n = 50$ and $x = 32$, yielding $\hat{p} = 0.64$. The number of future trials is $m = 50$. The 80 percent prediction interval is $50(0.64) \pm$

$$1.282\sqrt{50(0.64)(0.36)}\sqrt{1 + \frac{50}{50}} =$$

$$32 \pm 6.1 = [25.9, 38.1].$$

After rounding, the interval is [26, 38].

- (b) Yes, because the number of successes on Tuesday, 34, is within the interval.
- (c) Tuesday's probability of success is unknown. With the given assumption, Monday's data can be used to estimate Tuesday's p ; without the assumption, there is no data with which to estimate Tuesday's p .

9. 0.0210.

10. (a) The previous data gives $\hat{p} = 26/40 = 0.65$. Thus, the point prediction of the total number of successes is $m\hat{p} = 60(0.65) = 39$.
- (b) The prediction interval is $39 \pm$

$$1.645\sqrt{60(0.65)(0.35)}\sqrt{1 + 60/40} =$$

$$39 \pm 9.6 = [29.4, 48.6].$$

After rounding, the prediction interval is [29, 49].

- (c) Mieke actually obtained $y = 47$ successes. The point prediction (39) was incorrect, but the prediction interval was correct.
11. The center of a confidence interval for p always equals \hat{p} . The three given intervals have centers equal to 0.44, 0.41, and 0.41, respectively. Thus, the first interval must be Sam's. The last interval is Rachel's 90 percent interval because it is the narrower of her two intervals.
12. 0.073. (From Table A.4.)
13. The value of $p = 0.3038$ is known and $m = 1000$. First, compute

$$1.645\sqrt{1000(0.3038)(0.6962)} = 23.9.$$

Thus, the 90 percent prediction interval is

$$1000(0.3038) \pm 23.9 = [279.9, 327.7].$$

After rounding, the prediction interval is [280, 328].

14. False. (The value of \hat{p} is 0.700, but the exact value of p cannot be determined from data.)
15. The standardized test statistic is

$$z = \frac{x - np_0}{\sqrt{np_0q_0}} = \frac{120 - 100}{\sqrt{500(0.20)(0.80)}} = 2.24.$$

Thus, the approximate P-value equals $2(0.0125) = 0.0250$.