Chapter 5

One-Population Models

5.1 Study Suggestions

In the first four chapters of the text, the results of a study literally were restricted to the subjects in the study. Chapter 5 begins the investigation of methods for extending what has been learned in a study. The notion of a population is the fundamental concept involved in these extensions.

Throughout Chapter 5 and the remainder of the text, I stress the existence of two types of populations that correspond to the two types of subjects introduced in Chapter 1, distinct individuals or trials. At first my approach may appear to be a wasteful extravagance because the mathematical techniques that work for one type of population also work for the other type. There are a number of advantages, however, in having two types of populations; some of these advantages are discussed below.

- A finite population is a tangible entity. The population of students registered at my university this semester is very concrete. By contrast, an infinite population is a bit more slippery for a nonmathematician. The notion of "a mathematical model for the process that generates the outcomes of my shooting free throws," is a bit overwhelming for many of my students.

- A census is possible for a finite population, but not for an infinite population.

- Obtaining a random sample presents a different challenge for the two types of populations. For a finite population the challenge lies in obtaining a listing of the population members, selecting a sample of subjects at random, finding the selected subjects, and convincing them to participate in the study. For trials, the challenge lies in verifying (see below) the assumptions of Bernoulli trials.

- The goals of inference can depend on the type of population. For example, in many applications prediction is the most important method of inference for trials, but it is rarely important for a finite population.

For the material in Chapter 15, it is important to realize that the population in Chapter 5 is a single number p.

Pages 153–55 of the text provide a brute-force justification of the multiplication rule. This argument does not work well for my students and I no longer use it in my class. Instead, I appeal to the long-run relative frequency interpretation of probability to motivate the multiplication rule. More precisely, suppose $p = 0.6$, and suppose interest lies in the event

$$(X_1 = 1, X_2 = 0).$$

The condition for invoking the long-run relative frequency interpretation of probability is met: the experiment of selecting two cards at random with replacement from the population box can certainly be repeated a large number of times under identical conditions. Suppose that the experiment is, in fact, to be repeated a large number of times. Since $p = 0.6$, about 60 percent of the first selections will yield a card marked 1. Regardless of what happens on the first selection, because $q = 0.4$, about 40 percent of the second selections will yield a card marked 0. Hence, of the 60 percent of first selections that yield a 1, about 40 percent will yield a 0 on the second selection. Since, 40 percent of 60 percent is $0.4(0.6) = 0.24$, the multiplication rule has been "justified."

Section 5.3 on Bernoulli trials is unusual in its coverage. In addition to performing a mind experiment—thinking about whether the assumptions of Bernoulli trials are reasonable—you are encouraged to examine data in a critical manner. In particular, a few simple techniques are introduced that allow you to assess informally the validity of the assumptions of Bernoulli trials. Later, in Chapter 7, these assessments are made more formal.

5.2 Solutions to Odd-Numbered Exercises

Solutions for Section 5.2

3. (a) $0.3(0.3)(0.7)(0.7) = 0.0441.$
   (b) $0.3(0.7)(0.3)(0.7) = 0.0441.$
   (c) The stated event occurs if, and only if, either of the sequences 1,0,1,0 or 0,1,0,1 occur. As shown in part (b), the probability of the first of these sequences is 0.0441. By a similar argument the second of these sequences has the same probability. Thus, the probability of the indicated event is $0.0441 + 0.0441 = 0.0882.$

5. The multiplication rule can be used because the random variables are independent. Thus, the answer is $0.6(0.5) = 0.30.$

7. The random variable X has the binomial sampling distribution with $n = 2$ and $p = 0.65$. Thus,

   (a) $P(X = 0) =$
   $$\frac{2!}{0!2!}(0.65)^0(0.35)^2 = 0.1225.$$
   (b) $P(X = 1) =$
   $$\frac{2!}{1!1!}(0.65)^1(0.35)^1 = 0.4550.$$
   (c) $P(X = 2) =$
   $$\frac{2!}{2!0!}(0.65)^2(0.35)^0 = 0.4225.$$

9. The random variable X has the binomial sampling distribution with $n = 6$ and $p = 0.65$. Thus,

   (a) $P(X = 3) =$
   $$\frac{6!}{3!3!}(0.65)^3(0.35)^3 = 0.2355.$$
   (b) $P(X = 4) =$
   $$\frac{6!}{4!2!}(0.65)^4(0.35)^2 = 0.3280.$$

11. (a) $P(X = 3) =$
   $$\frac{6!}{3!3!}(0.4)^3(0.6)^3 = 0.2765.$$
   (b) $P(X = 4) =$
   $$\frac{5!}{4!1!}(0.7)^4(0.3)^1 = 0.3602.$$
   (c) $P(X = 2) =$
   $$\frac{7!}{2!5!}(0.2)^2(0.8)^5 = 0.2753.$$

Solutions for Section 5.3

1. Let X denote the student's score. Given the student is guessing, the answer to each question can be viewed as a Bernoulli trial with probability of success, a correct answer, equal to 0.5. The total number of correct answers, X, therefore has a binomial sampling distribution with $n = 25$ and $p = 0.5$. The event "the student passes," is identical to the event $(X \geq 15)$. Thus, the probability that the student passes equals $P(X \geq 15)$. Reading from Table 5.3 in the text, this probability is 0.2122.

3. Let X denote the student's score. Given the student is guessing, the answer to each question can be viewed as a Bernoulli trial with probability of success, a correct answer, equal to 0.25.

The total number of correct answers, X, therefore has a binomial sampling distribution with $n = 5$ and $p = 0.25$. The probability that the student scores one or more correct is most easily obtained by using the complement rule. Thus,

$$P(X \geq 1) = 1 - P(X = 0) = 1 - (0.75)^5 =$$

$$1 - 0.2373 = 0.7627.$$

5.3 Exam Questions

1. Al, Bev, Cy, Di, and Ed have identical boxes that contain 10 cards, numbered 1, 2, . . . , 10. Each person selects a card at random from his/her box. What is the probability that at least one of these 5 persons select a card marked 10?

2. Suppose Clyde Gaines is in the process of shooting 100 free throws, but his performance is deteriorating over time. If we consider each shot a trial, what Bernoulli trial assumption is most seriously violated?

   (a) Each trial results in one of two possible outcomes.

   (b) The probability of success remains constant from trial to trial.

   (c) The trials are independent.

3. I decide to shoot 100 free throws. The first free throw is attempted with my eyes open. For the remaining free throws I use the following protocol. Every successful shot is followed by a shot with my eyes closed, and every failed shot is followed by a shot with my eyes open. Assume that I am a much better shot with my eyes open than with my eyes closed. At the end of my 100 shots, I determine the entries in the following table:

Previous Trial	Current Trial S	F	Total
S			
F			
Total			

   Which value do you anticipate will be larger for this table, $\hat{p}_1$ or $\hat{p}_2$? Briefly justify your answer.

4. Sally enjoys shooting free throws. Assume that Sally's free throws satisfy the assumptions of Bernoulli Trials with $p = 0.8$. Compute the probability that her next three free throws yield two successes followed by a failure (in that order).

5. Cliff has developed a new way to "randomize" for a balanced design, and his method will be investigated in this exercise. Toss a fair coin for the first subject. If the coin lands heads, then the subject is assigned to treatment 1, but if it lands tails the subject is assigned to treatment 2. Repeat this process for the second subject. Continue to proceed in this way until either treatment has its full allotment of subjects. At that time, stop tossing the coin and assign the remaining subjects to achieve balance. For example, suppose $n = 4$ and the result of the coin tossing is HTT. Following Cliff's algorithm, the first subject is assigned to treatment 1 because the first toss of the coin yielded a head. Similarly, the second and third subjects are assigned to treatment 2. At this point, treatment 2 has its full allotment of subjects (two), and the remaining subject is assigned to treatment 1 to achieve balance.

   Norm is not convinced that Cliff's method "works," even if the coin is "fair." Help settle this dispute by computing the entries in the probability column of the following table. Comment on the importance of your answer.

Result of coin tossing	Subjects on Tr. 1	Subjects on Tr. 2	Probability
HH	1, 2	3, 4	
TT	3, 4	1, 2	
HTH	1, 3	2, 4	
THH	2, 3	1, 4	
HTT	1, 4	2, 3	
THT	2, 4	1, 3	

6. The Bin(10,0.5) sampling distribution is below.

x	$P(X = x)$	$P(X \leq x)$	$P(X \geq x)$
0	0.0010	0.0010	1.0000
1	0.0098	0.0107	0.9990
2	0.0439	0.0547	0.9893
3	0.1172	0.1719	0.9453
4	0.2051	0.3770	0.8281
5	0.2461	0.6230	0.6230
6	0.2051	0.8281	0.3770
7	0.1172	0.9453	0.1719
8	0.0439	0.9893	0.0547
9	0.0098	0.9990	0.0107
10	0.0010	1.0000	0.0010

Each of 50 persons tosses a fair coin 10 times. The person who obtains the most heads wins a prize. Suppose it turns out that Sam and Norm each obtain 8 heads and everyone else obtains 7 or fewer heads. Since the prize cannot be shared, it is decided that Sam and Norm will each toss the coin 10 more times and whoever achieves more heads will win the prize. Cliff opines that the increased pressure of this 'toss-off' will result in each player performing worse than he did in his original 10 tosses. What do you think about Cliff's prediction? (Your answer should include a computation of the probability of the event "both Sam and Norm perform worse" and a discussion of Cliff's conjecture about pressure.)

7. A box contains 100 cards. Sixty-five cards are marked '1' and the remaining 35 cards are marked '0.' Marge plans to select 20 cards at random with replacement from the box. Let X denote the sum of the numbers on the 20 selected cards.

   (a) Use the appropriate formula to compute $E(X)$.

   (b) True or false? The probability histogram of the sampling distribution for X is symmetric.

   (c) Write down the expression for $P(X = 12)$. Do not compute the answer, but replace all symbols with numbers.

8. For his class project Ben R. obtained data from the 58 *Black Crowes* concerts that were performed between January 9, 1995, and September 4, 1995. A concert was labeled a success if it included the three songs *My morning song*, *Wiser time*, and *Remedy*, and was labeled a failure if at least one of these songs was not played. You are given the following information about the data.

   - Eleven of the first 29 concerts were successes, and a total of 26 concerts were successes.
   - The first concert was a success and the last concert was a failure.
   - On 20 occasions a failure was followed by a success.

   Use the definitions and information given above to critically examine the conjecture that these 58 *Black Crowes* concerts satisfy the three assumptions of Bernoulli trials. Your answer should include appropriate tables and computations. It should also include an interpretation of your findings.

9. John is an archer and likes to shoot arrows at a target. Past experience suggests that it is reasonable to assume that John's shots are Bernoulli trials with a probability of success (hitting the target) equal to 0.85. Compute the probability that in the next four shots, John obtains exactly three successes.

10. Mieke performed 40 trials. Each trial was playing a concert B♭ on her clarinet into a tuner. The tuner then reports whether the note is sharp, flat, or perfectly in tune. Define *perfectly in tune* a success, and either *sharp* or *flat* a failure. Below are data from Mieke's study.

F	S	S	F	F	F	S	S	S	F
S	F	F	F	S	S	S	S	S	S
F	F	S	S	S	S	S	S	F	S
S	F	F	F	S	S	S	S	S	S

Clearly these are dichotomous trials (that is, each trial is a success or failure). Use the information given above to critically examine the conjecture that these 40 trials satisfy the remaining two assumptions of Bernoulli trials. Your answer should include appropriate tables and computations. It should also include an interpretation of your findings.

11. Denise performs 15 dichotomous trials (that is, each trial has a dichotomous response—either a success or a failure) and obtains the following data (in order, from left to right).

S S S F F S F F
S F S F S S F

Use this information to complete the following table.

Previous Trial	Current Trial S	Current Trial F	Total
S			
F			
Total			

12. Roger likes to make his canine companion Casey balance a treat on her muzzle until he gives her permission to eat it. Once permission has been given, Casey jerks her head to either side, causing the treat to fall. Casey then tries to catch the treat (a success) before it hits the ground (a failure). Casey is allowed to eat the treat only if she is successful in catching it. Roger performs an experiment consisting of 41 of these dichotomous trials, and Casey obtains 21 successes. In the course of analyzing his data, Roger obtains the following table.

Previous Trial	Current Trial S	Current Trial F	Total
S	a	b	20
F	c	d	20
Total	20	20	40

Match each of these four possible values of a, 5, 10, 11, and 19, with one statement given below.

A. There is no evidence that the outcome of the previous trial has any influence on the outcome of the current trial.

B. There is very weak evidence that Casey performs better after a success than after a failure.

C. There is very strong evidence that Casey performs better after a success than after a failure.

D. There is substantial evidence that Casey performs better after a failure than after a success.

13. Roger likes to make his canine companion Casey balance a treat on her muzzle until he gives her permission to eat it. Once permission has been given, Casey jerks her head to the side, causing the treat to fall. Casey then tries to catch the treat (a success) before it hits the ground (a failure). Casey is allowed to eat the treat only if she is successful in catching it. Roger performs an experiment consisting of 40 of these dichotomous trials, and Casey obtains 24 successes. In the course of analyzing his data, Roger obtains the following table.

	S	F	Total
First Half	a	b	20
Second Half	c	d	20
Total	24	16	40

Match each of these five possible values of a, 4, 11, 12, 13 and 20, with one statement given below.

A. There is very weak evidence that Casey's ability improved during the course of the study.

B. There is no evidence that Casey's ability changed during the course of the study.

C. There is very strong evidence that Casey's ability improved during the course of the study.

D. There is very strong evidence that Casey's ability declined during the course of the study.

E. There is very weak evidence that Casey's ability declined during the course of the study.

14. The collection of all current Wisconsin driver's license holders is an example of a(n)

    (a) Infinite population
    (b) Random sample
    (c) Finite population
    (d) Dichotomous box

15. Below is the Bin(10,0.5) sampling distribution.

x	$P(X = x)$	$P(X \leq x)$	$P(X \geq x)$
0	0.0010	0.0010	1.0000
1	0.0098	0.0107	0.9990
2	0.0439	0.0547	0.9893
3	0.1172	0.1719	0.9453
4	0.2051	0.3770	0.8281
5	0.2461	0.6230	0.6230
6	0.2051	0.8281	0.3770
7	0.1172	0.9453	0.1719
8	0.0439	0.9893	0.0547
9	0.0098	0.9990	0.0107
10	0.0010	1.0000	0.0010

    Alex plans to toss a fair coin 10 times, as do Bobby and Tony. Elaine wins $10 if all three of the men obtain six or fewer heads. Compute the probability that Elaine wins $10.

16. You are given the following facts.

    - $P(X = 0) = 0.30$ and $P(X = 1) = 0.70$.
    - The random variables X and Y are independent and identically distributed.

    What is $P(X = Y)$?

    (a) 0.30
    (b) 0.70
    (c) 0.58
    (d) 0.09
    (e) 0.49
    (f) Not enough information is provided to obtain an answer.

5.4 Solutions to Exam Questions

1. Let selecting a 10 denote a success and selecting anything else denote a failure. Let X equal the number of these five persons who obtain a success. Upon reflection, one realizes that X has a binomial distribution with $n = 5$ and $p = 0.1$. The question asks for $P(X \geq 1)$. By the complement rule, $P(X \geq 1) = 1 - P(X = 0)$. Finally, $P(X = 0) = (0.9)^5 = 0.5905$. Thus, the requested answer is $1 - 0.5905 = 0.4095$.

2. (b).

3. All of the shots in the first row of the table are attempted with my eyes closed, while all of the shots in the second row of the table are attempted with my eyes open. Thus, I anticipate the second row will have a (much) larger proportion of successes than will the first row.

4. $P(SSF) = 0.8(0.8)(0.2) = 0.128$.

5. The probabilities for the table, top to bottom, are 0.250, 0.250, 0.125, 0.125, 0.125, and 0.125.

    Cliff is mistaken because the six assignments are not equally likely.

6. The probability that Sam performs worse in the 'toss-off' is $P(X \leq 7) = 0.9453$. The same can be said for Norm. Thus, the probability they both perform worse is $(0.9453)^2 = 0.8936$.

    Norm and Sam were lucky in the first round. The odds are against either of them being so lucky again and pressure has nothing to do with it (since we are told they are tossing a fair coin).

7. Note that $p = 0.65$, $q = 0.35$, and $n = 20$.

    (a) $E(X) = 20(0.65) = 13$.
    (b) False.
    (c)

$$P(X = 12) = \frac{20!}{12!8!}(0.65)^{12}(0.35)^8.$$

8. The first assumption is satisfied because each trial has two possible outcomes. The second assumption is investigated by examining the following table.

Half	S	F	Total
First	11	18	29
Second	15	14	29
Total	26	32	58

This table has $\hat{p}_1 = 0.38$ and $\hat{p}_2 = 0.52$. Thus, there is evidence that the probability of success increased over time.

The third assumption is investigated by examining the following table.

Previous	Current Trial		
Trial	S	F	Total
S	5	21	26
F	20	11	31
Total	25	32	57

This table has $\hat{p}_1 = 0.19$ and $\hat{p}_2 = 0.65$. Thus, there is substantial evidence of memory in the system—65 percent of failures are followed by successes, but only 19 percent of successes are followed by successes.

9. Let X denote the total number of successes John obtains. It follows that X has the binomial sampling distribution with $n = 4$ and $p = 0.85$. The desired probability equals

$$P(X = 3) = \frac{n!}{x!(n-x)!} p^x q^{n-x},$$

with $x = 3$, $n - x = 1$, and $q = 0.15$. Thus,

$$P(X = 3) = \frac{4!}{3!1!} 0.85^3 0.15^1 = 0.3685.$$

10. The following table compares the first and second halves of Mieke's study.

Half	S	F	Total
First	12	8	20
Second	14	6	20
Total	26	14	40

The proportion of successes increased from $\hat{p}_1 = 0.60$ to $\hat{p}_2 = 0.70$, providing some evidence that p did not remain constant.

The table below provides some evidence of memory in the system.

	Current		
Previous	S	F	Total
S	19	6	25
F	7	7	14
Total	26	13	39

Seventy-six percent of successes were followed by successes, but only 50 percent of failures were followed by successes.

11. The completed table is below.

Previous Trial	Current Trial S	F	Total
S	3	5	8
F	4	2	6
Total	7	7	14

12. A–10; B–11; C–19; D–5.

13. A–11; B–12; C–4; D–20; E–13.

14. (c).

15. $(0.8281)^3 = 0.5679$.

16. The answer is (c) because

$$P(X = Y) =$$
$$P(X = Y = 0 \text{ or } X = Y = 1) =$$
$$P(X = 0, Y = 0) + P(X = 1, Y = 1) =$$
$$0.30(0.30) + 0.70(0.70) =$$
$$0.09 + 0.49 = 0.58.$$

5.5 More Mathematics

The formulas for the mean and variance of the binomial sampling distributions are given in the text; in this section I will derive them.

The mean of the sampling distribution is

$$\mu = \sum_x x P(X = x) =$$
$$\sum_x x C(n, x) p^x q^{n-x}.$$

The summand equals 0 if $x = 0$. For $x \geq 1$,

$$xC(n, x) = nC(n-1, x-1).$$

Thus,

$$\mu = \sum_x npC(n-1, x-1)p^{x-1}q^{n-x}.$$

Now, substitute $y = x - 1$ and $m = n - 1$ to get

$$\mu = np \sum_y C(m, y)p^y q^{m-y} = np,$$

as stated in the text.

Let X have a binomial sampling distribution.

$$E[(X(X-1)] = \sum x(x-1)P(X = x) =$$

$$\sum_x x(x-1)C(n, x)p^x q^{n-x}.$$

The summand equals 0 if $x = 0$ or $x = 1$. For $x \geq 2$, the summand equals

$$\sum_x n(n-1)p^2 C(n-2, x-2)p^{x-2}q^{n-x} =$$

$$n(n-1)p^2.$$

Substituting the above values into

$$\text{Var}(X) = E[X(X-1)] + \mu - \mu^2$$

and simplifying yields

$$\text{Var}(X) = n(n-1)p^2 + np - n^2p^2 = .$$

$$n^2p^2 - np^2 + np - n^2p^2 = np - np^2 =$$

$$np(1-p) = npq,$$

as stated in the text.