Chapter 16

Numerical Data from Two Sources

16.1 Study Suggestions

Section 16.1 extends the ideas and results of Chapters 1–3 to a numerical response. Note, in particular, that probabilities are induced by the process of randomization, not by the assumption that the data are obtained by selecting independent random samples from populations. For a dichotomous response there is a natural choice, $\hat{p}_1 - \hat{p}_2$, for the value of the test statistic. For a numerical response, however, the choice is not obvious. The difference of the sample means and the difference of the sample medians are two viable candidates for the test statistic. Chapter 16 only considers the difference of the sample means as the test statistic.

The exact sampling distribution of the test statistic can be obtained by brute force for a study with a small number of subjects. For a study with a larger number of subjects, a computer simulation experiment with 10,000 runs will provide a good estimate of the sampling distribution of the test statistic. If you do not have access to a computer, then the t-distribution can be used to obtain an approximate P-value.

Section 16.2 considers the three popular sets of assumptions for inference for the difference of population means.

Section 16.3 extends the results on the randomized pairs design in Chapter 4 to a numerical response. Note that, as in Section 16.1, probabilities are induced by the process of randomization, not by the assumption that the data are obtained by selecting a random sample from the population of differences.

Section 16.4 considers a population model for paired data. A random sample from a population of differences can be analyzed with the methods of Chapter 15.

16.2 Solutions to Odd-Numbered Exercises

Solutions for Sections 16.1 and 16.2

1. (a)

$$s_p^2 = \frac{10(25) + 10(49)}{20} = 37.$$

Thus,

$$s_p = \sqrt{37} = 6.083.$$

(b)

$$s_p^2 = \frac{1(25) + 19(49)}{20} = 47.8.$$

Thus,

$$s_p = \sqrt{47.8} = 6.914.$$

(c)

$$s_p^2 = \frac{19(25) + 1(49)}{20} = 26.2.$$

Thus,

$$s_p = \sqrt{26.2} = 5.119.$$

3. Note that the values of s_p for parts (a)–(c) were computed in parts (a)–(c), respectively, of Exercise 1.

(a)

$$t = \frac{30 - 25}{6.083\sqrt{1/10 + 1/10}} = 1.838.$$

(b)

$$t = \frac{30 - 25}{6.914\sqrt{1/2 + 1/20}} = 0.975.$$

(c)

$$t = \frac{30 - 25}{5.119\sqrt{1/20 + 1/2}} = 1.317.$$

5. (a) First, compute $s_p = 8.06$. Next, because the study is balanced,

$$t = \frac{\sqrt{n}(\bar{x} - \bar{y})}{2s_p} = \frac{\sqrt{20}(13.5)}{2(8.06)} = 3.75.$$

The approximate P-value for the third alternative is twice the area to the right of 3.75 under the t-curve with 18 degrees of freedom. This area is less than 0.01. Brian is faster when wearing jungle boots, and the difference is highly statistically significant.

(b) For 18 degrees of freedom, $t = 2.101$, and the 95 percent confidence interval is

$$(\bar{x} - \bar{y}) \pm ts_p\sqrt{1/n_1 + 1/n_2} =$$

$$13.5 \pm 2.101(8.06)\sqrt{1/10 + 1/10} =$$

$$13.5 \pm 7.6 = [5.9, 21.1].$$

At the 95 percent confidence level, one can conclude that Brian's mean time in jungle boots is between 5.9 and 21.1 seconds lower than his mean time in combat boots.

7. Note that because of the balance and the sample size, all three cases give the same answer.

(a) First, compute $s_p = 36.35$. Next, because the study is balanced,

$$t = \frac{\sqrt{n}(\bar{x} - \bar{y})}{2s_p} = \frac{\sqrt{80}(19.83)}{2(36.35)} = 2.44.$$

The approximate P-value for the first alternative is the area to the right of 2.44 under the standard normal curve. This area equals 0.0073. Jennifer hit the ball farther with the aluminum bat, and the difference is highly statistically significant.

(b) For 78 degrees of freedom, $t = z = 1.96$, and the 95 percent confidence interval is

$$(\bar{x} - \bar{y}) \pm ts_p\sqrt{1/n_1 + 1/n_2} =$$

$$19.83 \pm 1.96(36.35)\sqrt{1/40 + 1/40} =$$

$$19.83 \pm 15.93 = [2.90, 35.76].$$

At the 95 percent confidence level, one can conclude that Jennifer's mean distance with the aluminum bat is between 2.98 and 35.76 feet greater than her mean distance with the wooden bat.

9. Note that because of the balance and the sample size, all three cases give the same answer.

(a) First, compute $s_p = 29.11$. Next, because the study is balanced,

$$t = \frac{\sqrt{n}(\bar{x} - \bar{y})}{2s_p} = \frac{\sqrt{80}(8.69)}{2(29.11)} = 1.34.$$

The approximate P-value for the first alternative is the area to the right of 1.34 under the standard normal curve. This area equals 0.0901. The observed difference between the wood and the iron is not statistically significant.

(b) For 78 degrees of freedom, $t = z = 1.96$, and the 95 percent confidence interval is

$$(\bar{x} - \bar{y}) \pm ts_p\sqrt{1/n_1 + 1/n_2} =$$

$$8.69 \pm 1.96(29.11)\sqrt{1/40 + 1/40} =$$

$$8.69 \pm 12.76 = [-4.07, 21.45].$$

At the 95 percent confidence level, one can conclude that the mean for the wood is as much as 21.45 yards larger than the mean for the iron, or the mean for the iron is as much as 4.07 yards larger than the mean for the wood.

11. (a) First, compute $s_p = 0.3088$. Next, because the study is balanced,

$$t = \frac{\sqrt{n}(\bar{x} - \bar{y})}{2s_p} = \frac{\sqrt{30}(0.84)}{2(0.3088)} = 7.45.$$

The approximate P-value is the area to the right of 7.45 under the t-curve with 28 degrees of freedom. This area is smaller than 0.005. The data are highly statistically significant, and one can conclude that the mean acceleration time is larger when the windows are open than when the windows are closed.

(b) I will use case 1. Because the study is balanced, the sample sizes are not too small, and the values of the sample standard do not differ by much, case 2 would yield a similar answer. For 28 degrees of freedom, $t = 2.048$, and the 95 percent confidence interval is

$$(\bar{x} - \bar{y}) \pm ts_p\sqrt{1/n_1 + 1/n_2} =$$

$$0.84 \pm 2.048(0.3088)\sqrt{1/15 + 1/15} =$$

$$0.84 \pm 0.23 = [0.61, 1.07].$$

The population mean acceleration time with the windows open is, at the 95 percent confidence level, between 0.61 and 1.07 seconds smaller than the population mean acceleration time with the windows closed.

13. (a) First, compute $s_p = 8.55$. Next, because the study is balanced,

$$t = \frac{\sqrt{n}(\bar{x} - \bar{y})}{2s_p} = \frac{\sqrt{40}(8.0)}{2(8.55)} = 2.96.$$

The approximate P-value for the first alternative is the area to the right of 2.96 under the standard normal curve. This area equals 0.0015.

(b) For 38 degrees of freedom, $t = z = 1.96$, and the 95 percent confidence interval is

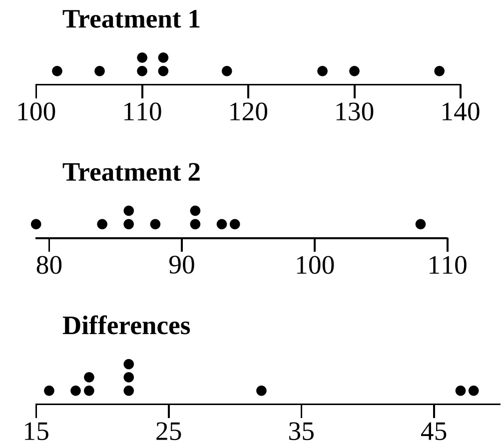
$$(\bar{x} - \bar{y}) \pm ts_p\sqrt{1/n_1 + 1/n_2} =$$

$$8.0 \pm 1.96(8.55)\sqrt{1/20 + 1/20} =$$

$$8.0 \pm 5.3 = [2.7, 13.3].$$

Solutions for Sections 16.3 and 16.4

1. (a) The dot plots are below. The scores on the first treatment are skewed to the right. Except for one large outlier, the distribution of scores on the second treatment is nearly symmetric. The differences have two gaps and two or three large outliers.



(b) For probabilities induced by the process of randomization, the statistician's answer is correct. (There are 2^{10} possible assignments of subjects to treatments, and obtaining differences that are all positive is clearly the most extreme possibility.) If one assumes that the differences are a random sample from a population of differences and that the population is a normal curve, then the t-distribution can be used to obtain a P-value. The assumption of a random sample might be reasonable—although we cannot check it with so little data. The assumption of a normal population, however, seems unreasonable in view of the dot plot of the differences. With such a small sample size, it does not seem reasonable to believe the answer obtained from the t-distribution is robust for this application.

(c) The test statistic is

$$t = \frac{\sqrt{n}(\bar{d})}{s_D} =$$

$$\frac{\sqrt{10}(26.5)}{11.87} = 7.06.$$

The approximate P-value for the first alternative is the area to the right of 7.06 under the t-curve with $n - 1 = 10 - 1 = 9$ degrees of freedom. This area is smaller than 0.005.

3. (a) The number of degrees of freedom is $n - 1 = 10 - 1 = 9$, yielding $t = 2.262$. Thus, the 95 percent confidence interval is

$$\bar{d} \pm t(\frac{s_D}{\sqrt{n}}) = 26.5 \pm 2.262(\frac{11.87}{\sqrt{10}}) =$$

$$26.5 \pm 8.5 = [18.0, 35.0].$$

(b) The 95 percent confidence interval is

$$\bar{x} \pm t(\frac{s_X}{\sqrt{n}}) = 116.5 \pm 2.262(\frac{11.56}{\sqrt{10}}) =$$

$$116.5 \pm 8.3 = [108.2, 124.8].$$

(c) The 95 percent confidence interval is

$$\bar{y} \pm t(\frac{s_Y}{\sqrt{n}}) = 90.0 \pm 2.262(\frac{7.77}{\sqrt{10}}) =$$

$$90.0 \pm 5.6 = [84.4, 95.6].$$

7. (a) The test statistic is

$$t = \frac{\sqrt{n}(\bar{d})}{s_D} = \frac{\sqrt{5}(20.4)}{8.17} = 5.58.$$

The approximate P-value for the first alternative is the area to the right of 5.58 under the t-curve with $n - 1 = 5 - 1 = 4$ degrees of freedom. This area is smaller than 0.005. (Note that because all of the differences are positive, the exact P-value for probabilities induced by randomization is $(0.5)^5 = 0.0312$.)

The observed difference between using two hands and one hand is highly statistically significant.

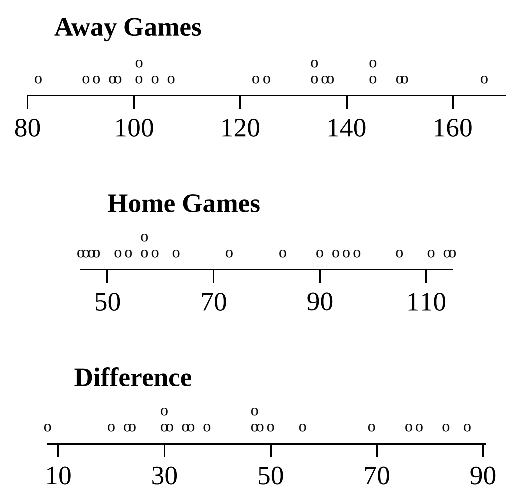
(b) The number of degrees of freedom is $n - 1 = 5 - 1 = 4$, yielding $t = 2.776$. Thus, the 95 percent confidence interval is

$$\bar{d} \pm t(\frac{s_D}{\sqrt{n}}) = 20.4 \pm 2.776(\frac{8.17}{\sqrt{5}}) =$$

$$20.4 \pm 10.1 = [10.3, 30.5].$$

At the 95 percent confidence level, the population mean score using two hands minus the population mean score using one hand is between 10.3 and 30.5 pins. This analysis assumes that the differences in the data set are the result of taking a random sample from the population that represents the process that generates the differences. This process is assumed to be stationary (it does not change), and the trials are assumed to be independent. The population is assumed to be a normal curve. In view of the small sample size, the t-distribution confidence interval might not be very robust for this application.

11. (a) The dot plots are below. There is a large amount of spread in each dot plot. There are no outliers, but it is difficult to discern any shape from these plots; perhaps a stem and leaf plot or histogram would work better.



(b) The number of degrees of freedom is $n - 1 = 20 - 1 = 19$, yielding $t = 2.093$. Thus, the 95 percent confidence interval is

$$\bar{d} \pm t(\frac{s_D}{\sqrt{n}}) = 45.7 \pm 2.093(\frac{22.74}{\sqrt{20}}) =$$

$$45.7 \pm 10.6 = [35.1, 56.3].$$

13. (a) The number of degrees of freedom is $n - 1 = 74 - 1 = 73$, yielding $t = z = 1.96$. Thus, the 95 percent confidence interval is

$$\bar{d} \pm t(\frac{s_D}{\sqrt{n}}) = 1.45 \pm 1.96(\frac{18.10}{\sqrt{74}}) =$$

$$1.45 \pm 4.12 = [-2.67, 5.57].$$

The confidence interval is correct because it includes 0.

(b) Because $n = 74$ is larger than 20, the approximate method must be used.

$$k' = \frac{75}{2} - \frac{1.96\sqrt{74}}{2} = 37.5 - 8.4 = 29.1.$$

Rounding down, $k = 29$ and the confidence interval is

$$[x_{(29)}, x_{(46)}] = [-2, 0].$$

16.3 Exam Questions

1. Chad performed a balanced CRD with 20 trials and a numerical response. A trial was street-skating 40 yards as fast as possible, the response was Chad's time to complete the trial, and the treatments were carrying and not carrying a hockey stick.

   Chad's data are reported in hundredths of seconds. For example, the response 479 means 4.79 seconds.

   Chad's sorted times with the stick are:

467	479	493	502	513
520	522	530	531	542

   Chad's sorted times without the stick are:

474	475	476	483	484
488	496	499	502	510

   The mean and standard deviation for the "with stick" data are 509.9 and 24.22; for the "without stick" data, these summaries are 488.7 and 12.52.

   Following the terminology of Chapter 16 of the text, assume case 1 for these data; that is, assume that the data are independent random samples, the populations are normal curves, and each population has the same standard deviation.

   (a) Test the null hypothesis that the two population means are equal versus the alternative that they are not equal.

   (b) Construct a 95 percent confidence interval for the difference of the population means.

   (c) Before collecting data, Chad believed that, on average (mean), he would be faster with the stick than without it. Use your answers to parts (a) or (b) above to comment on this belief.

   (d) Before collecting data, Sally believed that, on average (mean), Chad would be one-quarter of one second (0.25 seconds, or 25 hundredths of a second) faster without the stick than with it. Use your answers to parts (a) or (b) above to comment on this belief.

2. Refer to question 1. Obtain a confidence interval for the population median of Chad's times with the stick. Select your confidence level and remember to report it with your answer.

3. Refer to question 1. Obtain a 90 percent confidence interval for the population mean of Chad's times without the stick.

5. Bob performs a balanced CRD with 10 trials, two treatments, and a numerical response. Bob obtained $\bar{x} = 80$ and $\bar{y} = 61$. Bob wants to approximate the sampling distribution of the difference of sample means (treatment 1 minus treatment 2) for the null hypothesis of homogeneity of treatment effects.

   Bob performs a 1,000-run simulation experiment and obtains the following results:

- Five of the runs yield $\bar{x} = 80$.
- Eighty-seven of the runs yield $\bar{x} > 80$.
- Eighty-four of the runs yield $\bar{x} < 61$.
- Two of the runs yield $\bar{x} = 61$.

Compute Bob's simulation estimate of the P-value for the first alternative.

6. Refer to the previous question. Compute Bob's simulation estimate of the P-value for the third alternative.

7. Bob performs a balanced CRD with 10 trials, two treatments, and a numerical response. His data give $\bar{x} = 80$, $\bar{y} = 61$, $s_X = 23.1$, and $s_Y = 23.5$. Use the t-distribution to approximate the P-value for the first alternative for the test of homogeneity of treatment effects.

8. Refer to the previous question. Assume that the data are independent random samples from two populations. Under the assumptions of Case 1 in the text, construct a 95 percent confidence interval for $\mu_X - \mu_Y$.

16.4 Solutions to Exam Questions

1. (a) Compute

$$s_p^2 = \frac{(24.22)^2 + (12.52)^2}{2} = 371.68.$$

Thus, $s_p = \sqrt{371.68} = 19.28$. The test statistic is

$$t_1 = \frac{509.9 - 488.7}{19.28\sqrt{1/10 + 1/10}} =$$

$$\frac{21.2}{8.62} = 2.46.$$

The reference curve is the t-curve with 18 degrees of freedom. The area to the right of 2.46 under the reference curve is between 0.01 and 0.025. Thus, the P-value for the third alternative is between 0.02 and 0.05.

(b) The confidence interval is

$$21.2 \pm 2.101(19.28)\sqrt{1/10 + 1/10} =$$

$$21.2 \pm 18.1 = [3.1, 39.3].$$

(c) Part (a) indicates that the null hypothesis should be rejected, but part (b) indicates that Chad was actually faster without the stick.

(d) Sally's special value, 25 hundredths of a second, lies in the 95 percent confidence interval. Thus, the data do not contradict her belief.

2. There are three permissible answers: a 99.8 percent confidence interval is given by $[467, 542]$, a 97.9 percent confidence interval is given by $[479, 531]$, and a 89.1 percent confidence interval is given by $[493, 530]$.

3. The 90 percent confidence interval is

$$488.7 \pm 1.833(\frac{12.52}{\sqrt{10}}) =$$

$$488.7 \pm 7.3 = [481.4, 496.0].$$

5. 0.092.

6. 0.178.

7. $t = 1.29$ with eight degrees of freedom. Thus, $0.10 < \mathbf{P} < 0.25$.

8. The confidence interval is

$$19.0 \pm 34.0 = [-15.0, 53.0].$$