

# Lecture 21: Monotone likelihood ratio and UMP tests

## Monotone likelihood ratio

A simple hypothesis involves only one population.

If a hypothesis is not simple, it is called composite.

UMP tests for a composite  $H_1$  exist in Example 6.2.

We now extend this result to a class of parametric problems in which the likelihood functions have a special property.

### Definition 6.2

Suppose that the distribution of  $X$  is in  $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ , a parametric family indexed by a real-valued  $\theta$ , and that  $\mathcal{P}$  is dominated by a  $\sigma$ -finite measure  $\nu$ .

Let  $f_\theta = dP_\theta/d\nu$ .

The family  $\mathcal{P}$  is said to have *monotone likelihood ratio* in  $Y(X)$  (a real-valued statistic) if and only if, for any  $\theta_1 < \theta_2$ ,  $f_{\theta_2}(x)/f_{\theta_1}(x)$  is a nondecreasing function of  $Y(x)$  for values  $x$  at which at least one of  $f_{\theta_1}(x)$  and  $f_{\theta_2}(x)$  is positive.

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## Lemma 6.3

Suppose that the distribution of  $X$  is in a parametric family  $\mathcal{P}$  indexed by a real-valued  $\theta$  and that  $\mathcal{P}$  has monotone likelihood ratio in  $Y(X)$ . If  $\psi$  is a nondecreasing function of  $Y$ , then  $g(\theta) = E[\psi(Y)]$  is a nondecreasing function of  $\theta$ .

Take  $\psi(y) = I_{(t, \infty)}(y)$ .

Then  $g(\theta) = P(Y > t) = 1 - F_Y(t)$  is nondecreasing in  $\theta$ .

## Example 6.3

Let  $\theta$  be real-valued and  $\eta(\theta)$  be a nondecreasing function of  $\theta$ . Then the one-parameter exponential family with

$$f_{\theta}(x) = \exp\{\eta(\theta)Y(x) - \xi(\theta)\}h(x)$$

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## Example 6.4

Let  $X_1, \dots, X_n$  be i.i.d. from the uniform distribution on  $(0, \theta)$ , where  $\theta > 0$ .

The Lebesgue p.d.f. of  $X = (X_1, \dots, X_n)$  is  $f_\theta(x) = \theta^{-n} I_{(0, \theta)}(x_{(n)})$ , where  $x_{(n)}$  is the value of the largest order statistic  $X_{(n)}$ .

For  $\theta_1 < \theta_2$ ,

$$\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)} = \frac{\theta_1^n I_{(0, \theta_2)}(x_{(n)})}{\theta_2^n I_{(0, \theta_1)}(x_{(n)})},$$

which is a nondecreasing function of  $x_{(n)}$  for  $x$ 's at which at least one of  $f_{\theta_1}(x)$  and  $f_{\theta_2}(x)$  is positive, i.e.,  $x_{(n)} < \theta_2$ .

Hence the family of distributions of  $X$  has monotone likelihood ratio in  $X_{(n)}$ .

## Example 6.5

The following families have monotone likelihood ratio:

- the double exponential distribution family  $\{DE(\theta, c)\}$  with a known  $c$ ;
- the exponential distribution family  $\{E(\theta, c)\}$  with a known  $c$ ;
- the logistic distribution family  $\{LG(\theta, c)\}$  with a known  $c$ ;
- the uniform distribution family  $\{U(\theta, \theta + 1)\}$ ;
- the hypergeometric distribution family  $\{HG(r, \theta, N - \theta)\}$  with known  $r$  and  $N$  (Table 1.1, page 18).

An example of a family that does not have monotone likelihood ratio is the Cauchy distribution family  $\{C(\theta, c)\}$  with a known  $c$ .

## Testing one sided hypotheses

Hypotheses of the form  $H_0 : \theta \leq \theta_0$  (or  $H_0 : \theta \geq \theta_0$ ) versus  $H_1 : \theta > \theta_0$  (or  $H_1 : \theta < \theta_0$ ) are called *one-sided* hypotheses for any fixed constant  $\theta_0$ .

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## Theorem 6.2

Suppose that  $X$  has a distribution in  $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$  ( $\Theta \subset \mathcal{R}$ ) that has monotone likelihood ratio in  $Y(X)$ .

Consider the problem of testing  $H_0 : \theta \leq \theta_0$  versus  $H_1 : \theta > \theta_0$ , where  $\theta_0$  is a given constant.

(i) There exists a UMP test of size  $\alpha$ , which is given by

$$T_*(X) = \begin{cases} 1 & Y(X) > c \\ \gamma & Y(X) = c \\ 0 & Y(X) < c, \end{cases}$$

where  $c$  and  $\gamma$  are determined by  $\beta_{T_*}(\theta_0) = \alpha$ , and  $\beta_T(\theta) = E[T(X)]$  is the power function of a test  $T$ .

(ii)  $\beta_{T_*}(\theta)$  is strictly increasing for all  $\theta$ 's for which  $0 < \beta_{T_*}(\theta) < 1$ .

(iii) For any  $\theta < \theta_0$ ,  $T_*$  minimizes  $\beta_T(\theta)$  (the type I error probability of  $T$ ) among all tests  $T$  satisfying  $\beta_T(\theta_0) = \alpha$ .

(iv) Assume that  $P_\theta(f_\theta(X) = cf_{\theta_0}(X)) = 0$  for any  $\theta > \theta_0$  and  $c \geq 0$ , where  $f_\theta$  is the p.d.f. of  $P_\theta$ .

If  $T$  is a test with  $\beta_T(\theta_0) = \beta_{T_*}(\theta_0)$ , then for any  $\theta > \theta_0$ , either  $\beta_T(\theta) < \beta_{T_*}(\theta)$  or  $T = T_*$  a.s.  $P_\theta$ .

## Theorem 6.2 (continued)

(v) For any fixed  $\theta_1$ ,  $T_*$  is UMP for testing  $H_0 : \theta \leq \theta_1$  versus  $H_1 : \theta > \theta_1$ , with size  $\beta_{T_*}(\theta_1)$ .

### Remark

By reversing inequalities throughout, we can obtain UMP tests for testing  $H_0 : \theta \geq \theta_0$  versus  $H_1 : \theta < \theta_0$ .

### Proof of Theorem 6.2

(i) Consider the hypotheses  $\theta = \theta_0$  versus  $\theta = \theta_1$  with any  $\theta_1 > \theta_0$ . A UMP test is given in Theorem 6.1 with  $f_j =$  the p.d.f. of  $P_{\theta_j}$ ,  $j = 0, 1$ . Since  $\mathcal{P}$  has monotone likelihood ratio in  $Y(X)$ , this UMP test can be chosen to be the same as  $T_*$  with possibly different  $c$  and  $\gamma$  satisfying  $\beta_{T_*}(\theta_0) = \alpha$ .

Since  $T_*$  does not depend on  $\theta_1$ , it follows from Lemma 6.1 that  $T_*$  is UMP for testing the hypothesis  $\theta = \theta_0$  versus  $H_1$ .

## Theorem 6.2 (continued)

(v) For any fixed  $\theta_1$ ,  $T_*$  is UMP for testing  $H_0 : \theta \leq \theta_1$  versus  $H_1 : \theta > \theta_1$ , with size  $\beta_{T_*}(\theta_1)$ .

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## Proof (continued)

Note that if  $T_*$  is UMP for testing  $\theta = \theta_0$  versus  $H_1$ , then it is UMP for testing  $H_0$  versus  $H_1$ , provided that  $\beta_{T_*}(\theta) \leq \alpha$  for all  $\theta \leq \theta_0$ , i.e., the size of  $T_*$  is  $\alpha$ .

But this follows from Lemma 6.3, i.e.,  $\beta_{T_*}(\theta)$  is nondecreasing in  $\theta$ .

(ii) See Exercise 2 in §6.6.

(iii) The result can be proved using Theorem 6.1 with all inequalities reversed.

(iv) The proof for (iv) is left as an exercise.

(v) The proof for (v) is similar to that of (i).

## Corollary 6.1 (one-parameter exponential families)

Suppose that  $X$  has a p.d.f. in a one-parameter exponential family with  $\eta$  being a strictly monotone function of  $\theta$ . If  $\eta$  is increasing, then  $T_*$  given by Theorem 6.2 is UMP for testing  $H_0 : \theta \leq \theta_0$  versus  $H_1 : \theta > \theta_0$ , where  $\gamma$  and  $c$  are determined by  $\beta_{T_*}(\theta_0) = \alpha$ .

If  $\eta$  is decreasing or  $H_0 : \theta \geq \theta_0$  ( $H_1 : \theta < \theta_0$ ), the result is still valid by reversing inequalities in the definition of  $T_*$ .

## Proof (continued)

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## Example 6.6

Let  $X_1, \dots, X_n$  be i.i.d. from the  $N(\mu, \sigma^2)$  distribution with an unknown  $\mu \in \mathcal{R}$  and a known  $\sigma^2$ .

Consider  $H_0 : \mu \leq \mu_0$  versus  $H_1 : \mu > \mu_0$ , where  $\mu_0$  is a fixed constant. The p.d.f. of  $X = (X_1, \dots, X_n)$  is from a one-parameter exponential family with  $Y(X) = \bar{X}$  and  $\eta(\mu) = n\mu/\sigma^2$ .

By Corollary 6.1 and the fact that  $\bar{X}$  is  $N(\mu, \sigma^2/n)$ , the UMP test is  $T_*(X) = I_{(c_\alpha, \infty)}(\bar{X})$ , where  $c_\alpha = \sigma z_{1-\alpha}/\sqrt{n} + \mu_0$  and  $z_a = \Phi^{-1}(a)$ .

## Discussion

To derive a UMP test for testing  $H_0 : \theta \leq \theta_0$  versus  $H_1 : \theta > \theta_0$  when  $X$  has a p.d.f. in a one-parameter exponential family, it is essential to know the distribution of  $Y(X)$ .

Typically, a nonrandomized test can be obtained if the distribution of  $Y$  is continuous; otherwise UMP tests are randomized.

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## Example 6.8

Let  $X_1, \dots, X_n$  be i.i.d. random variables from the Poisson distribution  $P(\theta)$  with an unknown  $\theta > 0$ .

The p.d.f. of  $X = (X_1, \dots, X_n)$  is from a one-parameter exponential family with  $Y(X) = \sum_{i=1}^n X_i$  and  $\eta(\theta) = \log \theta$ .

Note that  $Y$  has the Poisson distribution  $P(n\theta)$ .

By Corollary 6.1, a UMP test for  $H_0 : \theta \leq \theta_0$  versus  $H_1 : \theta > \theta_0$  is given by Theorem 6.2 with  $c$  and  $\gamma$  satisfying

$$\alpha = \sum_{j=c+1}^{\infty} \frac{e^{n\theta_0} (n\theta_0)^j}{j!} + \gamma \frac{e^{n\theta_0} (n\theta_0)^c}{c!}.$$

## Example 6.9

Let  $X_1, \dots, X_n$  be i.i.d. random variables from the uniform distribution  $U(0, \theta)$ ,  $\theta > 0$ .

Consider the hypotheses  $H_0 : \theta \leq \theta_0$  and  $H_1 : \theta > \theta_0$ .

The p.d.f. of  $X = (X_1, \dots, X_n)$  is in a family with monotone likelihood ratio in  $Y(X) = X_{(n)}$  (Example 6.4).

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## Example 6.9 (continued)

By Theorem 6.2, a UMP test is  $T_*$ .

Since  $X_{(n)}$  has the Lebesgue p.d.f.  $n\theta^{-n}x^{n-1}I_{(0,\theta)}(x)$ , the UMP test  $T_*$  is nonrandomized and

$$\alpha = \beta_{T_*}(\theta_0) = \frac{n}{\theta_0^n} \int_c^{\theta_0} x^{n-1} dx = 1 - \frac{c^n}{\theta_0^n}.$$

Hence  $c = \theta_0(1 - \alpha)^{1/n}$ .

The power function of  $T_*$  when  $\theta > \theta_0$  is

$$\beta_{T_*}(\theta) = \frac{n}{\theta^n} \int_c^{\theta} x^{n-1} dx = 1 - \frac{\theta_0^n(1 - \alpha)}{\theta^n}.$$

In this problem, however, UMP tests are not unique.

(Note that the condition  $P_{\theta}(f_{\theta}(X) = cf_{\theta_0}(X)) = 0$  in Theorem 6.2(iv) is not satisfied.)

It can be shown (exercise) that the following test is also UMP with size  $\alpha$ :

$$T(X) = \begin{cases} 1 & X_{(n)} > \theta_0 \\ \alpha & X_{(n)} \leq \theta_0. \end{cases}$$