Stat 710: Mathematical Statistics Lecture 41

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Example 7.27 (Multiple comparison in one-way ANOVA models)

Consider the one-way ANOVA model

$$X_{ij} = N(\mu_i, \sigma^2), \quad j = 1, ..., n_i, i = 1, ..., m.$$

If the hypothesis $H_0 : \mu_1 = \cdots = \mu_m$ is rejected, one typically would like to compare μ_i's.

One way to compare μ_i's is to consider simultaneous confidence intervals for $\mu_i - \mu_j$, $1 \le i < j \le m$.

Since X_{ij}'s are independently normal, the sample means $\bar{X}_{i\cdot}$ are independently normal $N(\mu_i, \sigma^2/n_i)$, $i = 1, ..., m$, respectively, and they are independent of

$$SSR = \sum_{i=1}^{m} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{i\cdot})^2.$$

Consequently, $(\bar{X}_{i\cdot} - \bar{X}_{j\cdot})/\sqrt{v_{ij}}$ has the t-distribution t_{n-m}, $1 \le i < j \le m$, where $v_{ij} = (n_i^{-1} + n_j^{-1})SSR/(n-m)$.

Example 7.27 (continued)

For each (i,j), a confidence interval for $\mu_i - \mu_j$ with confidence coefficient $1-\alpha$ is

$$C_{ij,\alpha}(X) = [\bar{X}_{i\cdot} - \bar{X}_{j\cdot} - t_{n-m,\alpha/2}\sqrt{v_{ij}}, \bar{X}_{i\cdot} - \bar{X}_{j\cdot} + t_{n-m,\alpha/2}\sqrt{v_{ij}}],$$

where $t_{n-m,\alpha}$ is the $(1-\alpha)$th quantile of the t-distribution t_{n-m}. One can show that $C_{ij,\alpha}(X)$ is actually UMAU (exercise).

Bonferroni's level $1-\alpha$ simultaneous confidence intervals for $\mu_i - \mu_j$, $1 \le i < j \le m$, are $C_{ij,\alpha_*}(X)$, $1 \le i < j \le m$, where $\alpha_* = 2\alpha/[m(m-1)]$. When m is large, these confidence intervals are very conservative in the sense that the confidence coefficient of these intervals may be much larger than the nominal level $1-\alpha$ and these intervals may be too wide to be useful.

If the normality assumption is removed, then $C_{ij,\alpha}(X)$ is $1-\alpha$ asymptotically correct as $\min\{n_1,...,n_m\} \to \infty$ and $\max\{n_1,...,n_m\}/\min\{n_1,...,n_m\} \to c < \infty$.
Therefore, $C_{ij,\alpha_*}(X)$, $1 \le i < j \le m$, are simultaneous confidence intervals with asymptotic confidence level $1-\alpha$.

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Example 7.27 (continued)

We can also use Scheffé's method to obtain simultaneous confidence intervals for $\mu_i - \mu_j$, $1 \le i < j \le m$.

Scheffé's intervals have confidence coefficient $1-\alpha$ for $t^\tau L\beta$, $t \in \mathscr{T}$, where $\beta = (\mu_1, ..., \mu_m)$ and

$$L = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}.$$

Scheffé's intervals are also too conservative.

In fact, they are often more conservative than Bonferroni's intervals.

Tukey's method in one-way ANOVA models

Consider the one-way ANOVA model in Example 7.27.

Tukey's method introduced next produces simultaneous confidence intervals for all nonzero contrasts (including the differences $\mu_i - \mu_j$, $1 \le i < j \le m$) with confidence coefficient $1-\alpha$.

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Studentized range

Let $\widehat{\sigma}^2 = SSR/(n-m)$, where SSR is given in Example 7.27.
The *studentized range* is defined to be

$$R_{st} = \max_{1 \le i < j \le m} \frac{|(\bar{X}_{i\cdot} - \mu_i) - (\bar{X}_{j\cdot} - \mu_j)|}{\widehat{\sigma}}.$$

The distribution of R_{st} does not depend on any unknown parameter.

Theorem 7.11

Assume the one-way ANOVA model in Example 7.27.
Let q_α be the $(1-\alpha)$th quantile of the studentized range R_{st}.
Then Tukey's intervals

$$[c^\tau \widehat{\beta} - q_\alpha \widehat{\sigma} c_+, c^\tau \widehat{\beta} + q_\alpha \widehat{\sigma} c_+], \qquad c \in \mathscr{R}^m - \{0\}, c^\tau J = 0,$$

are simultaneous confidence intervals for $c^\tau \beta$, $c \in \mathscr{R}^m - \{0\}$, $c^\tau J = 0$, with confidence coefficient $1-\alpha$, where c_+ is the sum of all positive components of c, $\beta = (\mu_1, ..., \mu_m)$, $\widehat{\beta} = (\bar{X}_{1\cdot}, ..., \bar{X}_{m\cdot})$, and J is the m-vector of ones.

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Proof

Let $Y_i = (\bar{X}_{i\cdot} - \mu_i)/\widehat{\sigma}$ and $Y = (Y_1, ..., Y_m)$.
Then the result follows if we can show that

$$\max_{1 \le i < j \le m} |Y_i - Y_j| \le q_\alpha \tag{1}$$

is equivalent to

$$|c^\tau Y| \le q_\alpha c_+ \qquad \text{for all } c \in \mathscr{R}^m \text{ satisfying } c^\tau J = 0, c \neq 0. \tag{2}$$

Let $c(i,j) = (c_1, ..., c_m)$ with $c_i = 1$, $c_j = -1$, and $c_l = 0$ for $l \neq i$ or $l \neq j$.
Then

$$c(i,j)_+ = 1 \qquad \text{and} \qquad |[c(i,j)]^\tau Y| = |Y_i - Y_j|.$$

Therefore, (2) implies (1).

Next, we show (1) implies (2).
Let $c = (c_1, ..., c_m)$ be a vector satisfying the conditions in (2).
Define $-c_-$ to be the sum of negative components of c.
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Proof (continued)

$$
\begin{aligned}
|c^\tau Y| &= \frac{1}{c_+}\left| c_+ \sum_{j:c_j<0} c_j Y_j + c_- \sum_{i:c_i>0} c_i Y_i \right| \\
&= \frac{1}{c_+}\left| \sum_{i:c_i>0} \sum_{j:c_j<0} c_i c_j Y_j - \sum_{j:c_j<0} \sum_{i:c_i>0} c_i c_j Y_i \right| \\
&= \frac{1}{c_+}\left| \sum_{i:c_i>0} \sum_{j:c_j<0} c_i c_j (Y_j - Y_i) \right| \\
&\leq \frac{1}{c_+} \sum_{i:c_i>0} \sum_{j:c_j<0} |c_i c_j||Y_j - Y_i| \\
&\leq \max_{1\leq i<j\leq m} |Y_j - Y_i| \left(\frac{1}{c_+} \sum_{i:c_i>0} \sum_{j:c_j<0} |c_i||c_j| \right) \\
&= \max_{1\leq i<j\leq m} |Y_j - Y_i| c_+,
\end{aligned}
$$

where the first and the last equalities follow from the fact that $c_- = c_+ \neq 0$.

Remarks

- Tukey's method works well when n_i's are all equal to n_0, in which case values of $\sqrt{n_0}q_\alpha$ can be found using tables or statistical software.
- When n_i's are unequal, some modifications are suggested; see Tukey (1977) and Milliken and Johnson (1992).

Example 7.29

We compare the t-type confidence intervals, Bonferroni's, Scheffé's, and Tukey's simultaneous confidence intervals for $\mu_i - \mu_j$, $1 \le i < j \le 3$, based on the following data X_{ij} given in Mendenhall and Sincich (1995):

	$j=1$	2	3	4	5	6	7	8	9	10
$i=1$	148	76	393	520	236	134	55	166	415	153
2	513	264	433	94	535	327	214	135	280	304
3	335	643	216	536	128	723	258	380	594	465

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Example 7.29 (continued)

In this example, $m = 3$, $n_i \equiv n_0 = 10$, $\bar{X}_{1\cdot} = 229.6$, $\bar{X}_{2\cdot} = 309.8$, $\bar{X}_{3\cdot} = 427.8$, and $\widehat{\sigma} = 168.95$.

Let $\alpha = 0.05$.

For the t-type intervals, $t_{27,0.975} = 2.05$.

For Bonferroni's method, $\alpha_* = \alpha/3 = 0.017$ and $t_{27,0.983} = 2.55$.

For Scheffé's method, $c_{0.05} = 3.35$ and $\sqrt{2c_{0.05}} = 2.59$.

From Table 13 in Mendenhall and Sincich (1995, Appendix II), $\sqrt{n_0}q_{0.05} = 3.49$.

The resulting confidence intervals are given in the following table.

Example 7.29 (continued)

Method	Parameter $\mu_1 - \mu_2$	$\mu_1 - \mu_3$	$\mu_2 - \mu_3$	Length
t-type	$[-235.2, 74.6]$	$[-353.1, -43.3]$	$[-272.8, 37.0]$	309.8
Bonferroni	$[-273.0, 112.4]$	$[-390.9, -5.5]$	$[-310.6, 74.8]$	385.4
Scheffé	$[-276.0, 115.4]$	$[-393.9, -2.5]$	$[-313.6, 77.8]$	391.4
Tukey	$[-267.3, 106.7]$	$[-385.2, -11.2]$	$[-304.9, 69.1]$	374.0

Discussions

- Apparently, t-type intervals have the shortest length, but they are not simultaneous confidence intervals.
- Tukey's intervals in this example have the shortest length among simultaneous confidence intervals.
- Scheffé's intervals have the longest length.