Stat 710: Mathematical Statistics
Lecture 39

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Lecture 39: Asymptotic confidence sets and quantiles

We consider another example of asymptotic confidence sets based on likelihood discussed in the last lecture.

Example 7.24

Let $X_1,...,X_n$ be i.i.d. from $N(\mu,\varphi)$ with unknown $\theta = (\mu,\varphi)$.
Consider the problem of constructing a $1-\alpha$ asymptotically correct confidence set for θ.
The log-likelihood function is

$$\log \ell(\theta) = -\frac{1}{2\varphi}\sum_{i=1}^{n}(X_i-\mu)^2 - \frac{n}{2}\log\varphi - \frac{n}{2}\log(2\pi).$$

Since $(\bar{X},\widehat{\varphi})$ is the MLE of θ, where $\widehat{\varphi} = (n-1)S^2/n$, the confidence set based on LR tests is

$$C_1(X) = \left\{\theta : \frac{1}{\varphi}\sum_{i=1}^{n}(X_i-\mu)^2 + n\log\varphi \le \chi^2_{2,\alpha} + n + n\log\widehat{\varphi}\right\}.$$

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Example 7.24 (continued)

Note that

$$s_n(\theta) = \left(\frac{n(\bar{X}-\mu)}{\varphi}, \frac{1}{2\varphi^2}\sum_{i=1}^{n}(X_i-\mu)^2 - \frac{n}{2\varphi} \right) \qquad I_n(\theta) = \begin{pmatrix} \frac{n}{\varphi} & 0 \\ 0 & \frac{n}{2\varphi^2} \end{pmatrix}.$$

Hence, the confidence set based on Wald's tests is

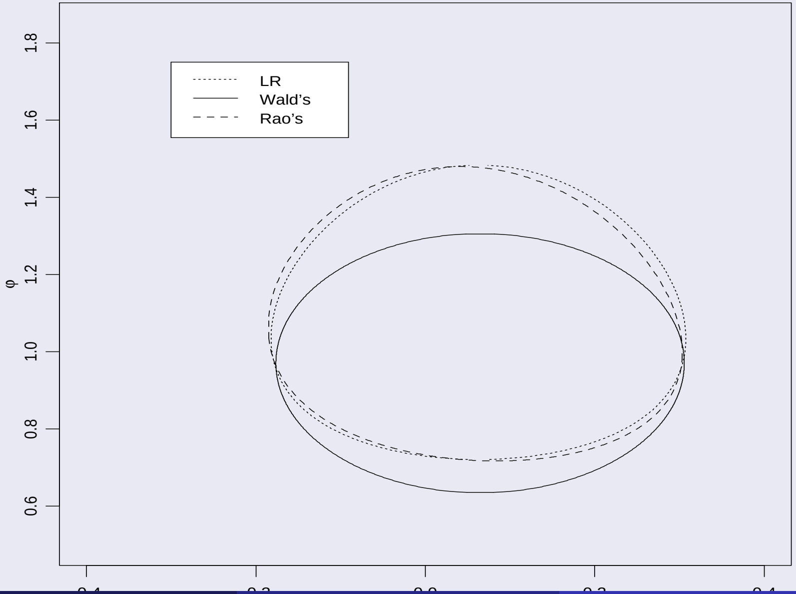
$$C_2(X) = \left\{ \theta : \frac{(\bar{X}-\mu)^2}{\widehat{\varphi}} + \frac{(\widehat{\varphi}-\varphi)^2}{2\widehat{\varphi}^2} \le \frac{\chi^2_{2,\alpha}}{n} \right\},$$

which is an ellipsoid in $\mathscr{R}^2$, and the confidence set based on Rao's score tests is

$$C_3(X) = \left\{ \theta : \frac{(\bar{X}-\mu)^2}{\varphi} + \frac{1}{2}\left[\frac{1}{n\varphi}\sum_{i=1}^{n}(X_i-\mu)^2 - 1 \right]^2 \le \frac{\chi^2_{2,\alpha}}{n} \right\}.$$

In general, $C_j(X)$, $j = 1, 2, 3$, are different.
An example of these three confidence sets is given in Figure 7.2, where $n = 100$, $\mu = 0$, and $\varphi = 1$.

Figure 7.2. Confidence sets obtained by inverting LR, Wald's, and Rao's score tests in Example 7.24



Example 7.24 (continued)

Consider now the construction of a confidence set for μ.

It can be shown (exercise) that the confidence set based on Wald's tests is defined by $C_2(X)$ with φ replaced by $\widehat{\varphi}$, whereas the confidence sets based on LR tests and Rao's score tests are defined by $C_1(X)$ and $C_3(X)$, respectively, with φ replaced by $n^{-1}\sum_{i=1}^n (X_i - \mu)^2$.

Confidence intervals for quantiles

Let $X_1, ..., X_n$ be i.i.d. from a continuous c.d.f. F on $\mathscr{R}$ and let $\theta = F^{-1}(p)$ be the pth quantile of F, $0 < p < 1$.

The general methods we previously discussed can be applied to obtain a confidence set for θ, but we introduce here a method that works particularly for quantile problems.

In fact, for any given α, it is possible to derive a confidence interval (or bound) for θ with confidence coefficient $1 - \alpha$ (Exercise 84), but the computation of such a confidence interval may be cumbersome.

We focus on asymptotic confidence intervals for θ.

Our result is based on the following result due to Bahadur (1966).

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Our result is based on the following result due to Bahadur (1966).

Theorem 7.8

Let $X_1,...,X_n$ be i.i.d. from a continuous c.d.f. F on $\mathscr{R}$ that is twice differentiable at $\theta = F^{-1}(p)$, $0 < p < 1$, with $F'(\theta) > 0$.
Let F_n be the empirical c.d.f.
Let $\{k_n\}$ be a sequence of integers satisfying $1 \le k_n \le n$ and $k_n/n = p + o\big((\log n)^\delta/\sqrt{n}\big)$ for some $\delta > 0$.
Then

$$X_{(k_n)} = \theta + \frac{(k_n/n) - F_n(\theta)}{F'(\theta)} + O\left(\frac{(\log n)^{(1+\delta)/2}}{n^{3/4}}\right) \text{ a.s.}$$

Proof

Omitted.

The result in Theorem 7.8 is a refinement of the Bahadur representation in Theorem 5.11.
The following corollary of Theorem 7.8 is useful in statistics.
Let $\widehat{\theta}_n = F_n^{-1}(p)$ be the sample pth quantile.

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Corollary 7.1

Assume the conditions in Theorem 7.8 and $k_n/n = p + cn^{-1/2} + o(n^{-1/2})$ with a constant c.
Then

$$\sqrt{n}(X_{(k_n)} - \widehat{\theta}_n) \to_{a.s.} c/F'(\theta).$$

Proof

Left as an exercise.

Using Corollary 7.1, we can obtain a confidence interval for θ with limiting confidence coefficient $1-\alpha$ (Definition 2.14) for any given $\alpha \in (0, \frac{1}{2})$.
This is stated and proved in the next result.

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Corollary 7.2

Assume the conditions in Theorem 7.8.
Let $\{k_{1n}\}$ and $\{k_{2n}\}$ be two sequences of integers satisfying $1 \le k_{1n} < k_{2n} \le n$,

$$k_{1n}/n = p - z_{1-\alpha/2}\sqrt{p(1-p)/n} + o(n^{-1/2}),$$

and

$$k_{2n}/n = p + z_{1-\alpha/2}\sqrt{p(1-p)/n} + o(n^{-1/2}),$$

where $z_a = \Phi^{-1}(a)$. Then the confidence interval $C(X) = [X_{(k_{1n})}, X_{(k_{2n})}]$ has the property that $P(\theta \in C(X))$ does not depend on P and

$$\lim_{n\to\infty} \inf_{P\in\mathscr{P}} P\big(\theta \in C(X)\big) = \lim_{n\to\infty} P\big(\theta \in C(X)\big) = 1-\alpha.$$

Furthermore,

$$\text{the length of } C(X) = \frac{2z_{1-\alpha/2}\sqrt{p(1-p)}}{F'(\theta)\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right) \quad \text{a.s.}$$

Proof

Note that $P(\theta \in C(X)) = P(X_{(k_{1n})} \le \theta \le X_{(k_{2n})}) = P(U_{(k_{1n})} \le p \le U_{(k_{2n})})$, where $U_{(k)}$ is the kth order statistic based on a sample $U_1,...,U_n$ i.i.d. from the uniform distribution $U(0,1)$ (Exercise 84).
Hence, $P(\theta \in C(X))$ does not depend on P and
$\lim_{n\to\infty} P\big(\theta \in C(X)\big) = \lim_{n\to\infty} \inf_{P\in\mathscr{P}} P\big(\theta \in C(X)\big)$.
By Corollary 7.1, Theorem 5.10, and Slutsky's theorem,

$$\begin{aligned} P(X_{(k_{1n})} > \theta) &= P\left(\widehat{\theta}_n - z_{1-\alpha/2}\frac{\sqrt{p(1-p)}}{F'(\theta)\sqrt{n}} + o_p(n^{-1/2}) > \theta\right) \\ &= P\left(\frac{\sqrt{n}(\widehat{\theta}_n - \theta)}{\sqrt{p(1-p)}/F'(\theta)} + o_p(1) > z_{1-\alpha/2}\right) \\ &\to 1 - \Phi(z_{1-\alpha/2}) \\ &= \alpha/2. \end{aligned}$$

The first result follows, since similarly $P(X_{(k_{2n})} < \theta) \to \alpha/2$.
The result for the length of $C(X)$ follows directly from Corollary 7.1.

Remarks

- The confidence interval $[X_{(k_{1n})}, X_{(k_{2n})}]$ given in Corollary 7.2 is called Woodruff's (1952) interval.
- It has limiting confidence coefficient $1-\alpha$, a property that is stronger than the $1-\alpha$ asymptotic correctness.
- The length of Woodruff's interval is $X_{(k_{2n})} - X_{(k_{1n})}$.
  By the result in Corollary 7.2,

$$X_{(k_{2n})} - X_{(k_{1n})} = \frac{2z_{\alpha/2}\sqrt{p(1-p)}}{\sqrt{n}F'(\theta)} + o\left(\frac{1}{\sqrt{n}}\right) \quad \text{a.s.,}$$

  This means

$$\frac{[X_{(k_{2n})} - X_{(k_{1n})}]^2}{4z^2_{\alpha/2}} = \frac{p(1-p)}{n[F'(\theta)]^2} + o\left(\frac{1}{n}\right) \quad \text{a.s.}$$

  Therefore, $[X_{(k_{2n})} - X_{(k_{1n})}]^2/(4z^2_{\alpha/2})$ is a consistent estimator of the asymptotic variance of the sample pth quantile.

Remarks

- From Theorem 5.10, if $F'(\theta)$ exists and is positive, then

$$\sqrt{n}(\widehat{\theta}_n - \theta) \to_d N\left(0, \frac{p(1-p)}{[F'(\theta)]^2}\right).$$

If the derivative $F'(\theta)$ has a consistent estimator $\widehat{d}_n$ obtained using some method such as one of those introduced in §5.1.3, then $\widehat{V}_n = p(1-p)/\widehat{d}_n^2$ is a consistent estimator of $p(1-p)/[F'(\theta)]^2$ and the method introduced in §7.3.1 can be applied to derive the following $1-\alpha$ asymptotically correct confidence interval:

$$C_1(X) = \left[\widehat{\theta}_n - z_{1-\alpha/2}\frac{\sqrt{p(1-p)}}{\widehat{d}_n\sqrt{n}}, \widehat{\theta}_n + z_{1-\alpha/2}\frac{\sqrt{p(1-p)}}{\widehat{d}_n\sqrt{n}}\right].$$

The length of $C_1(X)$ is asymptotically almost the same as Woodruff's interval.
However, $C_1(X)$ depends on the estimated derivative $\widehat{d}_n$ and it is usually difficult to obtain a precise estimator $\widehat{d}_n$.