

Stat 710: Mathematical Statistics

Lecture 37

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Lecture 37: Randomized confidence sets

Randomization

Applications of Theorems 7.4 and 7.5 require that $C(X)$ be obtained by inverting acceptance regions of nonrandomized tests.

Thus, these results cannot be directly applied to discrete problems. In fact, in discrete problems inverting acceptance regions of randomized tests may not lead to a confidence set with a given confidence coefficient.

Randomization is used in hypothesis testing to obtain tests with a given size.

Thus, the same idea can be applied to confidence sets, i.e., we may consider *randomized* confidence sets.

Inverting acceptance regions of randomized tests

Suppose that we invert acceptance regions of randomized tests T_{θ_0} that reject $H_0 : \theta = \theta_0$ with probability $T_{\theta_0}(x)$ when $X = x$.

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Inverting acceptance regions of randomized tests

Let U be a random variable that is independent of X and has the uniform distribution $U(0, 1)$.

Then the test $\tilde{T}_{\theta_0}(X, U) = I_{(U, 1]}(T_{\theta_0})$ has the same power function as T_{θ_0} and is "nonrandomized" if U is viewed as part of the sample.

Let

$$A_U(\theta_0) = \{(x, U) : U \geq T_{\theta_0}(x)\}$$

be the acceptance region of $\tilde{T}_{\theta_0}(X, U)$.

If T_{θ_0} has size α for all θ_0 , then inverting $A_U(\theta)$ we obtain a confidence set

$$C(X, U) = \{\theta : (X, U) \in A_U(\theta)\}$$

having confidence coefficient $1 - \alpha$, since

$$P(\theta \in C(X, U)) = E[P(U \geq T_{\theta}(X)|X)] = E[1 - T_{\theta}(X)].$$

If T_{θ_0} is UMP (or UMPU) for each θ_0 , then $C(X, U)$ is UMA (or UMAU). However, $C(X, U)$ is a randomized confidence set since it is still random when we observe $X = x$.

Integer valued statistic

When T_{θ_0} is a function of an integer-valued statistic, we can use the following method to derive $C(X, U)$.

Example 7.16

Let X_1, \dots, X_n be i.i.d. binary random variables with $p = P(X_i = 1)$. The confidence coefficient of $(\underline{p}, 1]$ may not be $1 - \alpha$, where \underline{p} is given by

$$\underline{p} = \inf\{p : m(p) \geq y\} = \inf\left\{p : \sum_{j=y}^n \binom{n}{j} p^j (1-p)^{n-j} \geq \alpha\right\}$$

(Example 7.7).

From Example 6.2 and the previous discussion, a randomized UMP test for testing $H_0 : p = p_0$ versus $H_1 : p > p_0$ can be constructed based on $Y = \sum_{i=1}^n X_i$ and U , a random variable that is independent of Y and has the uniform distribution $U(0, 1)$.

Since Y is integer-valued and $U \in (0, 1)$, $W = Y + U$ is equivalent to (Y, U) .

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Example 7.16 (continued)

For $w > 0$,

$$\begin{aligned}P(W \leq w) &= \sum_{j=0}^{\infty} P(W \leq w, Y = j) \\&= \sum_{j=0}^{\infty} P(Y = j)P(U \leq w - j) \\&= \sum_{j=0}^n \binom{n}{j} p^j (1 - p)^{n-j} (w - j) I_{(j, j+1]}(w),\end{aligned}$$

Thus, W has the following Lebesgue p.d.f.:

$$\begin{aligned}f_p(w) &= \sum_{j=0}^n \binom{n}{j} p^j (1 - p)^{n-j} I_{(j, j+1]}(w) \\&= \binom{n}{[w]} p^{[w]} (1 - p)^{n-[w]} I_{(0, n+1)}(w),\end{aligned}$$

where $[w]$ is the integer part of w .

Example 7.16 (continued)

For $p_1 < p_2$,

$$\frac{f_{p_2}(w)}{f_{p_1}(w)} = \frac{p_2^{[w]}(1-p_2)^{n-[w]}}{p_1^{[w]}(1-p_1)^{n-[w]}}$$

is increasing in $[w]$ and, hence, increasing in w , i.e.,

the family $\{f_p : p \in (0, 1)\}$ has monotone likelihood ratio in W .

It follows from Theorem 6.2 that the test $\tilde{T}_{p_0}(Y, U) = I_{(c(p_0), n+1)}(W)$ is UMP of size α for testing $H_0 : p = p_0$ versus $H_1 : p > p_0$, where

$$\alpha = \int_{c(p_0)}^{n+1} f_{p_0}(w) dw.$$

Then, $W \leq c(p_0)$ if and only if

$$\int_W^{n+1} f_p(w) dw \geq \int_{c(p_0)}^{n+1} f_{p_0}(w) dw = \alpha,$$

or

$$\int_0^W f_p(w) dw \leq 1 - \alpha.$$

Example 7.16 (continued)

Note that

$$\begin{aligned}\int_0^W f_p(w) dw &= \sum_{j=0}^{Y-1} \int_j^{j+1} f_p(w) dw + \int_Y^{Y+U} f_p(w) dw \\ &= \sum_{j=0}^{Y-1} \binom{n}{j} p^j (1-p)^{n-j} + U \binom{n}{Y} p^Y (1-p)^{n-Y} \\ &= F_p(Y-1) + U \binom{n}{Y} p^Y (1-p)^{n-Y},\end{aligned}$$

where $F_p(j)$ is the c.d.f. of Y .

From Lemma 6.3, $\int_0^W f_p(w) dw$ is decreasing in p .

Hence, inverting the acceptance regions of $\tilde{T}_p(Y, U)$ leads to

$$C(X, U) = \left\{ p : \int_0^W f_p(w) dw \leq 1 - \alpha \right\} = [\underline{p}_1, 1],$$

Example 7.16 (continued)

where \underline{p}_1 is the solution of

$$\int_0^W f_p(w) dw = F_p(Y-1) + U\psi_p(Y) = 1 - \alpha.$$

Consider first $0 < Y < n$.

Then

$$\binom{n}{Y} p^Y (1-p)^{n-Y} \rightarrow 0 \quad \text{as } p \rightarrow 0 \text{ or } 1$$

and

$$F_p(Y-1) = \sum_{j=0}^{Y-1} \binom{n}{j} p^j (1-p)^{n-j} = (1-p)^n + \sum_{j=1}^{Y-1} \binom{n}{j} p^j (1-p)^{n-j}$$

which $\rightarrow 0$ if $p \rightarrow 1$ and $\rightarrow 1$ if $p \rightarrow 0$.

Hence, $F_p(Y-1) + U\psi_p(Y) = 1 - \alpha$ has a unique solution, which is \underline{p}_1 .

Example 7.16 (continued)

When $Y = 0$, $F_p(Y - 1) = 0$.

Setting $F_p(Y - 1) + U(1 - p)^n = 1 - \alpha$ we obtain a solution

$$p = 1 - \left(\frac{1 - \alpha}{U} \right)^{1/n}.$$

Hence,

$$\underline{p}_1 = \begin{cases} 1 - \left(\frac{1 - \alpha}{U} \right)^{1/n} & U > 1 - \alpha \\ 0 & U \leq 1 - \alpha. \end{cases}$$

Similarly, when $Y = n$, $F_p(Y - 1) = 1 - p^n$.

Setting $F_p(Y - 1) + Up^n = 1 - \alpha$ we obtain a solution

$$p = \left(\frac{\alpha}{1 - U} \right)^{1/n}.$$

Hence,

$$\underline{p}_1 = \begin{cases} \left(\frac{\alpha}{1 - U} \right)^{1/n} & U < 1 - \alpha \\ 1 & U \geq 1 - \alpha. \end{cases}$$

Example 7.16 (continued)

The lower confidence bound \underline{p}_1 has confidence coefficient $1 - \alpha$ and is Θ' -UMA with $\Theta' = (0, p)$.

We can show that $\underline{p}_1 \geq \underline{p}$, where

$$\underline{p} = \inf \left\{ p : \sum_{j=y}^n \binom{n}{j} p^j (1-p)^{n-j} \geq \alpha \right\}$$

This is because

$$\underline{p} = \inf \{ p : F_p(Y-1) \leq 1 - \alpha \}.$$

Since \underline{p}_1 is the solution of $F_p(Y-1) + U\psi_p(Y) = 1 - \alpha$,

$$F_{\underline{p}_1}(Y-1) \leq F_{\underline{p}_1}(Y-1) + U\psi_{\underline{p}_1}(Y) = 1 - \alpha.$$

Hence, $\underline{p}_1 \geq \underline{p}$.

Remarks

- Although we considered the binomial distribution in Example 7.16, most of the discussions are valid when Y takes nonnegative integer values and its family has monotone likelihood ratio.
- Using a randomized confidence set, we can achieve the purpose of obtaining a confidence set with a given confidence coefficient as well as some optimality properties such as UMA, UMAU, or shortest expected length.
- On the other hand, randomization may not be desired in practical problems.