

# Stat 710: Mathematical Statistics

## Lecture 36

Jun Shao

Department of Statistics  
University of Wisconsin  
Madison, WI 53706, USA

## Confidence sets related to optimal tests

For a confidence set obtained by inverting the acceptance regions of some UMP or UMPU tests, it is expected that the confidence set inherits some optimality property.

### Definition 7.2

Let  $\theta \in \Theta$  be an unknown parameter and  $\Theta'$  be a subset of  $\Theta$  that does not contain the true parameter value  $\theta$ .

A confidence set  $C(X)$  for  $\theta$  with confidence coefficient  $1 - \alpha$  is said to be  $\Theta'$ -uniformly most accurate (UMA) iff for any other confidence set  $C_1(X)$  with confidence level  $1 - \alpha$ ,

$$P(\theta' \in C(X)) \leq P(\theta' \in C_1(X)) \quad \text{for all } \theta' \in \Theta'.$$

$C(X)$  is UMA iff it is  $\Theta'$ -UMA with  $\Theta' = \{\theta\}^c$ .

# Lecture 36: UMA and UMAU confidence sets

## Confidence sets related to optimal tests

For a confidence set obtained by inverting the acceptance regions of some UMP or UMPU tests, it is expected that the confidence set inherits some optimality property.

### Definition 7.2

Let  $\theta \in \Theta$  be an unknown parameter and  $\Theta'$  be a subset of  $\Theta$  that does not contain the true parameter value  $\theta$ .

A confidence set  $C(X)$  for  $\theta$  with confidence coefficient  $1 - \alpha$  is said to be  $\Theta'$ -uniformly most accurate (UMA) iff for any other confidence set  $C_1(X)$  with confidence level  $1 - \alpha$ ,

$$P(\theta' \in C(X)) \leq P(\theta' \in C_1(X)) \quad \text{for all } \theta' \in \Theta'.$$

$C(X)$  is UMA iff it is  $\Theta'$ -UMA with  $\Theta' = \{\theta\}^c$ .

## Remarks

- $P(\theta' \in C(X))$  is a probability of covering a false parameter value.
- Intuitively, confidence sets with small probabilities of covering wrong parameter values are preferred.
- The reason why we sometimes need to consider a  $\Theta'$  different from  $\{\theta\}^c$  (the set containing all false values) is that for some confidence sets, such as one-sided confidence intervals, we do not need to worry about the probabilities of covering some false values.
- For example, if we consider a lower confidence bound for a real-valued  $\theta$ , we are asserting that  $\theta$  is larger than a certain value and we only need to worry about covering values of  $\theta$  that are too small.  
Thus,  $\Theta' = \{\theta' \in \Theta : \theta' < \theta\}$ .
- A similar discussion leads to the consideration of  $\Theta' = \{\theta' \in \Theta : \theta' > \theta\}$  for upper confidence bounds.

## Theorem 7.4

Let  $C(X)$  be a confidence set for  $\theta$  obtained by inverting the acceptance regions of nonrandomized tests  $T_{\theta_0}$  for testing  $H_0 : \theta = \theta_0$  versus  $H_1 : \theta \in \Theta_{\theta_0}$ .

Suppose that for each  $\theta_0$ ,  $T_{\theta_0}$  is UMP of size  $\alpha$ .

Then  $C(X)$  is  $\Theta'$ -UMA with confidence coefficient  $1 - \alpha$ , where  $\Theta' = \{\theta' : \theta \in \Theta_{\theta'}\}$ .

## Proof

The fact that  $C(X)$  has confidence coefficient  $1 - \alpha$  follows from Theorem 7.2.

Let  $C_1(X)$  be another confidence set with confidence level  $1 - \alpha$ . By Proposition 7.2, the test

$$T_{1\theta_0}(X) = 1 - I_{A_1(\theta_0)}(X)$$

with  $A_1(\theta_0) = \{x : \theta_0 \in C_1(x)\}$  has significance level  $\alpha$  for testing  $H_0 : \theta = \theta_0$  versus  $H_1 : \theta \in \Theta_{\theta_0}$ .

## Theorem 7.4

Let  $C(X)$  be a confidence set for  $\theta$  obtained by inverting the acceptance regions of nonrandomized tests  $T_{\theta_0}$  for testing  $H_0 : \theta = \theta_0$  versus  $H_1 : \theta \in \Theta_{\theta_0}$ .

Suppose that for each  $\theta_0$ ,  $T_{\theta_0}$  is UMP of size  $\alpha$ .

Then  $C(X)$  is  $\Theta'$ -UMA with confidence coefficient  $1 - \alpha$ , where  $\Theta' = \{\theta' : \theta \in \Theta_{\theta'}\}$ .

## Proof

The fact that  $C(X)$  has confidence coefficient  $1 - \alpha$  follows from Theorem 7.2.

Let  $C_1(X)$  be another confidence set with confidence level  $1 - \alpha$ . By Proposition 7.2, the test

$$T_{1\theta_0}(X) = 1 - I_{A_1(\theta_0)}(X)$$

with  $A_1(\theta_0) = \{x : \theta_0 \in C_1(x)\}$  has significance level  $\alpha$  for testing  $H_0 : \theta = \theta_0$  versus  $H_1 : \theta \in \Theta_{\theta_0}$ .

## Proof (continued)

For any  $\theta' \in \Theta'$ ,  $\theta \in \Theta_{\theta'}$ , i.e.,  $P$  is in the family defined by  $H_1 : \theta \in \Theta_{\theta'}$ . Thus,

$$\begin{aligned} P(\theta' \in C(X)) &= 1 - P(T_{\theta'}(X) = 1) \\ &\leq 1 - P(T_{1\theta'}(X) = 1) \\ &= P(\theta' \in C_1(X)), \end{aligned}$$

where the first equality follows from the fact that  $T_{\theta'}$  is nonrandomized and the inequality follows from the fact that  $T_{\theta'}$  is UMP.

## Discussions

Theorem 7.4 can be applied to construct UMA confidence bounds in problems where the population is in a one-parameter parametric family with monotone likelihood ratio so that UMP tests exist (Theorem 6.2). It can also be applied to a few cases to construct two-sided UMA confidence intervals.

For example,  $[X_{(n)}, \alpha^{-1/n}X_{(n)}]$  in Example 7.13 is UMA.

## Proof (continued)

For any  $\theta' \in \Theta'$ ,  $\theta \in \Theta_{\theta'}$ , i.e.,  $P$  is in the family defined by  $H_1 : \theta \in \Theta_{\theta'}$ . Thus,

$$\begin{aligned} P(\theta' \in C(X)) &= 1 - P(T_{\theta'}(X) = 1) \\ &\leq 1 - P(T_{1\theta'}(X) = 1) \\ &= P(\theta' \in C_1(X)), \end{aligned}$$

where the first equality follows from the fact that  $T_{\theta'}$  is nonrandomized and the inequality follows from the fact that  $T_{\theta'}$  is UMP.

## Discussions

Theorem 7.4 can be applied to construct UMA confidence bounds in problems where the population is in a one-parameter parametric family with monotone likelihood ratio so that UMP tests exist (Theorem 6.2). It can also be applied to a few cases to construct two-sided UMA confidence intervals.

For example,  $[X_{(n)}, \alpha^{-1/n}X_{(n)}]$  in Example 7.13 is UMA.

As we discussed in §6.2, in many problems there are UMPU tests but not UMP tests.

### Definition 7.3

Let  $\theta \in \Theta$  be an unknown parameter,  $\Theta'$  be a subset of  $\Theta$  that does not contain the true parameter value  $\theta$ , and  $1 - \alpha$  be a given confidence level.

(i) A level  $1 - \alpha$  confidence set  $C(X)$  is said to be  $\Theta'$ -unbiased (unbiased when  $\Theta' = \{\theta\}^c$ ) iff

$$P(\theta' \in C(X)) \leq 1 - \alpha$$

for all  $\theta' \in \Theta'$ .

(ii) Let  $C(X)$  be a  $\Theta'$ -unbiased confidence set with confidence coefficient  $1 - \alpha$ . If

$$P(\theta' \in C(X)) \leq P(\theta' \in C_1(X)) \quad \text{for all } \theta' \in \Theta'.$$

holds for any other  $\Theta'$ -unbiased confidence set  $C_1(X)$  with confidence level  $1 - \alpha$ , then  $C(X)$  is  $\Theta'$ -uniformly most accurate unbiased (UMAU).  $C(X)$  is UMAU if and only if it is  $\Theta'$ -UMAU with  $\Theta' = \{\theta\}^c$ .

As we discussed in §6.2, in many problems there are UMPU tests but not UMP tests.

### Definition 7.3

Let  $\theta \in \Theta$  be an unknown parameter,  $\Theta'$  be a subset of  $\Theta$  that does not contain the true parameter value  $\theta$ , and  $1 - \alpha$  be a given confidence level.

(i) A level  $1 - \alpha$  confidence set  $C(X)$  is said to be  $\Theta'$ -unbiased (unbiased when  $\Theta' = \{\theta\}^c$ ) iff

$$P(\theta' \in C(X)) \leq 1 - \alpha$$

for all  $\theta' \in \Theta'$ .

(ii) Let  $C(X)$  be a  $\Theta'$ -unbiased confidence set with confidence coefficient  $1 - \alpha$ . If

$$P(\theta' \in C(X)) \leq P(\theta' \in C_1(X)) \quad \text{for all } \theta' \in \Theta'.$$

holds for any other  $\Theta'$ -unbiased confidence set  $C_1(X)$  with confidence level  $1 - \alpha$ , then  $C(X)$  is  $\Theta'$ -uniformly most accurate unbiased (UMAU).  $C(X)$  is UMAU if and only if it is  $\Theta'$ -UMAU with  $\Theta' = \{\theta\}^c$ .

## Theorem 7.5

Let  $C(X)$  be a confidence set for  $\theta$  obtained by inverting the acceptance regions of nonrandomized tests  $T_{\theta_0}$  for testing  $H_0 : \theta = \theta_0$  versus  $H_1 : \theta \in \Theta_{\theta_0}$ .

If  $T_{\theta_0}$  is unbiased of size  $\alpha$  for each  $\theta_0$ , then  $C(X)$  is  $\Theta'$ -unbiased with confidence coefficient  $1 - \alpha$ , where  $\Theta' = \{\theta' : \theta \in \Theta_{\theta'}\}$ .

If  $T_{\theta_0}$  is also UMPU for each  $\theta_0$ , then  $C(X)$  is  $\Theta'$ -UMAU.

## Example 7.15

Consider the normal linear model in Example 7.9 and the parameter  $\theta = l^r \beta$ , where  $l \in \mathcal{R}(Z)$ .

From §6.2.3, the nonrandomized test with acceptance region

$$A(\theta_0) = \left\{ X : l^r \hat{\beta} - \theta_0 > t_{n-r, \alpha} \sqrt{l^r (Z^r Z)^{-1} SSR / (n-r)} \right\}$$

is UMPU with size  $\alpha$  for testing  $H_0 : \theta = \theta_0$  versus  $H_1 : \theta < \theta_0$ , where  $\hat{\beta}$  is the LSE of  $\beta$  and  $t_{n-r, \alpha}$  is the  $(1 - \alpha)$ th quantile of the t-distribution  $t_{n-r}$ .

## Theorem 7.5

Let  $C(X)$  be a confidence set for  $\theta$  obtained by inverting the acceptance regions of nonrandomized tests  $T_{\theta_0}$  for testing  $H_0 : \theta = \theta_0$  versus  $H_1 : \theta \in \Theta_{\theta_0}$ .

If  $T_{\theta_0}$  is unbiased of size  $\alpha$  for each  $\theta_0$ , then  $C(X)$  is  $\Theta'$ -unbiased with confidence coefficient  $1 - \alpha$ , where  $\Theta' = \{\theta' : \theta \in \Theta_{\theta'}\}$ .

If  $T_{\theta_0}$  is also UMPU for each  $\theta_0$ , then  $C(X)$  is  $\Theta'$ -UMAU.

## Example 7.15

Consider the normal linear model in Example 7.9 and the parameter  $\theta = l^T \beta$ , where  $l \in \mathcal{R}(Z)$ .

From §6.2.3, the nonrandomized test with acceptance region

$$A(\theta_0) = \left\{ X : l^T \hat{\beta} - \theta_0 > t_{n-r, \alpha} \sqrt{l^T (Z^T Z)^{-1} \text{SSR} / (n-r)} \right\}$$

is UMPU with size  $\alpha$  for testing  $H_0 : \theta = \theta_0$  versus  $H_1 : \theta < \theta_0$ , where  $\hat{\beta}$  is the LSE of  $\beta$  and  $t_{n-r, \alpha}$  is the  $(1 - \alpha)$ th quantile of the t-distribution  $t_{n-r}$ .

## Example 7.15 (continued)

Inverting  $A(\theta)$  we obtain the following  $\Theta'$ -UMAU upper confidence bound with confidence coefficient  $1 - \alpha$  and  $\Theta' = (\theta, \infty)$ :

$$\bar{\theta} = l^r \hat{\beta} - t_{n-r, \alpha} \sqrt{l^r (Z^r Z)^{-1} SSR / (n-r)}.$$

A UMAU confidence interval for  $\theta$  can be similarly obtained.

## Exercise 7.60(a)

Let  $X_1$  and  $X_2$  be independent random variables from the exponential distributions on  $(0, \infty)$  with scale parameters  $\theta_1$  and  $\theta_2$ , respectively. We want to show that  $[\alpha Y / (2 - \alpha), (2 - \alpha) Y / \alpha]$  is a UMAU confidence interval for  $\theta_2 / \theta_1$  with confidence coefficient  $1 - \alpha$ , where  $Y = X_2 / X_1$ . First, we need to find a UMPU test of size  $\alpha$  for testing  $H_0 : \theta_2 = \lambda \theta_1$  versus  $H_1 : \theta_2 \neq \lambda \theta_1$ , where  $\lambda > 0$  is a known constant.

The joint density of  $X_1$  and  $X_2$  is

$$\frac{1}{\theta_1 \theta_2} \exp \left\{ -\frac{X_1}{\theta_1} - \frac{X_2}{\theta_2} \right\},$$

## Example 7.15 (continued)

Inverting  $A(\theta)$  we obtain the following  $\Theta'$ -UMAU upper confidence bound with confidence coefficient  $1 - \alpha$  and  $\Theta' = (\theta, \infty)$ :

$$\bar{\theta} = I^r \hat{\beta} - t_{n-r, \alpha} \sqrt{I^r (Z^r Z)^{-1} SSR / (n - r)}.$$

A UMAU confidence interval for  $\theta$  can be similarly obtained.

## Exercise 7.60(a)

Let  $X_1$  and  $X_2$  be independent random variables from the exponential distributions on  $(0, \infty)$  with scale parameters  $\theta_1$  and  $\theta_2$ , respectively. We want to show that  $[\alpha Y / (2 - \alpha), (2 - \alpha) Y / \alpha]$  is a UMAU confidence interval for  $\theta_2 / \theta_1$  with confidence coefficient  $1 - \alpha$ , where  $Y = X_2 / X_1$ . First, we need to find a UMPU test of size  $\alpha$  for testing  $H_0 : \theta_2 = \lambda \theta_1$  versus  $H_1 : \theta_2 \neq \lambda \theta_1$ , where  $\lambda > 0$  is a known constant.

The joint density of  $X_1$  and  $X_2$  is

$$\frac{1}{\theta_1 \theta_2} \exp \left\{ -\frac{X_1}{\theta_1} - \frac{X_2}{\theta_2} \right\},$$

## Exercise 7.60(a)

which can be written as

$$\frac{1}{\theta_1 \theta_2} \exp \left\{ -X_1 \left( \frac{1}{\theta_1} - \frac{\lambda}{\theta_2} \right) - (\lambda X_1 + X_2) \frac{1}{\theta_2} \right\}.$$

Hence, by Theorem 6.4, a UMPU test of size  $\alpha$  rejects  $H_0$  when  $X_1 < c_1(U)$  or  $X_1 > c_2(U)$ , where  $U = \lambda X_1 + X_2$ .

Note that  $X_1/X_2$  is independent of  $U$  under  $H_0$ .

Hence, by Lemma 6.7, the UMPU test is equivalent to the test that rejects  $H_0$  when  $X_1/X_2 < d_1$  or  $X_1/X_2 > d_2$ , which is equivalent to the test that rejects  $H_0$  when  $W < b_1$  or  $W > b_2$ , where  $W = \frac{Y/\lambda}{1+Y/\lambda}$  and  $b_1$  and  $b_2$  satisfy  $P(b_1 < W < b_2) = 1 - \alpha$  (for size  $\alpha$ ) and  $E[W I_{(b_1, b_2)}(W)] = (1 - \alpha)E(W)$  (for unbiasedness) under  $H_0$ .

When  $\theta_2 = \lambda \theta_1$ ,  $W$  has the same distribution as  $\frac{Z_1/Z_2}{1+Z_1/Z_2}$ , where  $Z_1$  and  $Z_2$  are independent and identically distributed random variables having the exponential distribution on  $(0, \infty)$  with scale parameter 1.

Hence, the distribution of  $W$  under  $H_0$  is uniform on  $(0, 1)$ .

## Exercise 7.60(a)

Then the requirements on  $b_1$  and  $b_2$  become  $b_2 - b_1 = 1 - \alpha$  and  $b_2^2 - b_1^2 = 1 - \alpha$ , which yield  $b_1 = \alpha/2$  and  $b_2 = 1 - \alpha/2$  (assuming that  $0 < \alpha < \frac{1}{2}$ ).

Hence, the acceptance region of the UMPU test is

$$A(\lambda) = \left\{ W : \frac{\alpha}{2} \leq W \leq 1 - \frac{\alpha}{2} \right\} = \left\{ Y : \frac{\alpha}{2 - \alpha} \leq \frac{Y}{\lambda} \leq \frac{2 - \alpha}{\alpha} \right\}.$$

Inverting  $A(\lambda)$  leads to

$$\{ \lambda : \lambda \in A(\lambda) \} = \left[ \frac{\alpha Y}{2 - \alpha}, \frac{(2 - \alpha) Y}{\alpha} \right],$$

which is a UMAU confidence interval for  $\lambda = \theta_2/\theta_1$  with confidence coefficient  $1 - \alpha$ .

The volume of a confidence set  $C(X)$  for  $\theta \in \mathcal{R}^k$  when  $X = x$  is defined to be

$$\text{vol}(C(x)) = \int_{C(x)} d\theta',$$

which is the Lebesgue measure of the set  $C(x)$  and may be infinite. In particular, if  $\theta$  is real-valued and  $C(X) = [\underline{\theta}(X), \bar{\theta}(X)]$  is a confidence interval, then  $\text{vol}(C(x))$  is simply the length of  $C(x)$ .

The next result reveals a relationship between the expected volume (length) and the probability of covering a false value of a confidence set (interval).

### Theorem 7.6 (Pratt's theorem)

Let  $X$  be a sample from  $P$  and  $C(X)$  be a confidence set for  $\theta \in \mathcal{R}^k$ . Suppose that  $\text{vol}(C(x)) = \int_{C(x)} d\theta'$  is finite a.s.  $P$ .

Then the expected volume of  $C(X)$  is

$$E[\text{vol}(C(X))] = \int_{\theta \neq \theta'} P(\theta' \in C(X)) d\theta'.$$

The volume of a confidence set  $C(X)$  for  $\theta \in \mathcal{R}^k$  when  $X = x$  is defined to be

$$\text{vol}(C(x)) = \int_{C(x)} d\theta',$$

which is the Lebesgue measure of the set  $C(x)$  and may be infinite. In particular, if  $\theta$  is real-valued and  $C(X) = [\underline{\theta}(X), \bar{\theta}(X)]$  is a confidence interval, then  $\text{vol}(C(x))$  is simply the length of  $C(x)$ .

The next result reveals a relationship between the expected volume (length) and the probability of covering a false value of a confidence set (interval).

### Theorem 7.6 (Pratt's theorem)

Let  $X$  be a sample from  $P$  and  $C(X)$  be a confidence set for  $\theta \in \mathcal{R}^k$ . Suppose that  $\text{vol}(C(x)) = \int_{C(x)} d\theta'$  is finite a.s.  $P$ .

Then the expected volume of  $C(X)$  is

$$E[\text{vol}(C(X))] = \int_{\theta \neq \theta'} P(\theta' \in C(X)) d\theta'.$$

## Proof

By Fubini's theorem,

$$\begin{aligned} E[\text{vol}(C(X))] &= \int \text{vol}(C(X)) dP \\ &= \int \left[ \int_{C(x)} d\theta' \right] dP(x) \\ &= \int \int_{\theta' \in C(x)} d\theta' dP(x) \\ &= \int \left[ \int_{\theta' \in C(x)} dP(x) \right] d\theta' \\ &= \int P(\theta' \in C(X)) d\theta' \\ &= \int_{\theta \neq \theta'} P(\theta' \in C(X)) d\theta'. \end{aligned}$$

This proves the result.

## Discussions

It follows from Theorem 7.6 that if  $C(X)$  is UMA (or UMAU) with confidence coefficient  $1 - \alpha$ , then it has the smallest expected volume among all confidence sets (or all unbiased confidence sets) with confidence level  $1 - \alpha$ .

For example, the confidence interval in Example 7.14 (when  $\sigma^2$  is known) or  $[X_{(n)}, \alpha^{-1/n}X_{(n)}]$  in Example 7.13 has the shortest expected length among all confidence intervals with confidence level  $1 - \alpha$ .