Stat 710: Mathematical Statistics
Lecture 33

Jun Shao

Department of Statistics
University of Wisconsin
Madison, WI 53706, USA

Lecture 33: Inverting acceptance regions of tests

Confidence sets and hypothesis tests

Another popular method of constructing confidence sets is to use a close relationship between confidence sets and hypothesis tests.
For any test T, the set $\{x : T(x) \neq 1\}$ is called the *acceptance region*.
This terminology is not precise when T is a randomized test.

Theorem 7.2

For each $\theta_0 \in \Theta$, let T_{θ_0} be a test for $H_0 : \theta = \theta_0$ (versus some H_1) with significance level α and acceptance region $A(\theta_0)$.
For each x in the range of X, define

$$C(x) = \{\theta : x \in A(\theta)\}.$$

Then $C(X)$ is a level $1 - \alpha$ confidence set for θ.
If T_{θ_0} is nonrandomized and has size α for every θ_0, then $C(X)$ has confidence coefficient $1 - \alpha$.

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Proof

We prove the first assertion only.
The proof for the second assertion is similar.
Under the given condition,

$$\sup_{\theta=\theta_0} P\big(X \notin A(\theta_0)\big) = \sup_{\theta=\theta_0} P(T_{\theta_0} = 1) \le \alpha,$$

which is the same as

$$1-\alpha \le \inf_{\theta=\theta_0} P\big(X \in A(\theta_0)\big) = \inf_{\theta=\theta_0} P\big(\theta_0 \in C(X)\big).$$

Since this holds for all θ_0, the result follows from

$$\inf_{P\in\mathscr{P}} P\big(\theta \in C(X)\big) = \inf_{\theta_0\in\Theta} \inf_{\theta=\theta_0} P\big(\theta_0 \in C(X)\big) \ge 1-\alpha.$$

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The converse of Theorem 7.2 is partially true.

Proposition 7.2

Let $C(X)$ be a confidence set for θ with confidence level (or confidence coefficient) $1-\alpha$.
For any $\theta_0 \in \Theta$, define a region $A(\theta_0) = \{x : \theta_0 \in C(x)\}$.
Then the test $T(X) = 1 - I_{A(\theta_0)}(X)$ has significance level α for testing $H_0 : \theta = \theta_0$ versus some H_1.

Discussions

In general, $C(X)$ in Theorem 7.2 can be determined numerically, if it does not have an explicit form.
Suppose $A(\theta) = \{Y : a(\theta) \le Y \le b(\theta)\}$ for a real-valued θ and statistic $Y(X)$ and some nondecreasing functions $a(\theta)$ and $b(\theta)$.
When we observe $Y = y$, $C(X)$ is an interval with limits $\underline{\theta}$ and $\overline{\theta}$, which are the θ-values at which the horizontal line $Y = y$ intersects the curves $Y = b(\theta)$ and $Y = a(\theta)$ (Figure 7.1), respectively.
If $y = b(\theta)$ (or $y = a(\theta)$) has no solution or more than one solution, $\underline{\theta} = \inf\{\theta : y \le b(\theta)\}$ (or $\overline{\theta} = \sup\{\theta : a(\theta) \le y\}$).
$C(X)$ does not include $\underline{\theta}$ (or $\overline{\theta}$) if and only if at $\underline{\theta}$ (or $\overline{\theta}$), $b(\theta)$ (or $a(\theta)$) is only left-continuous (or right-continuous).

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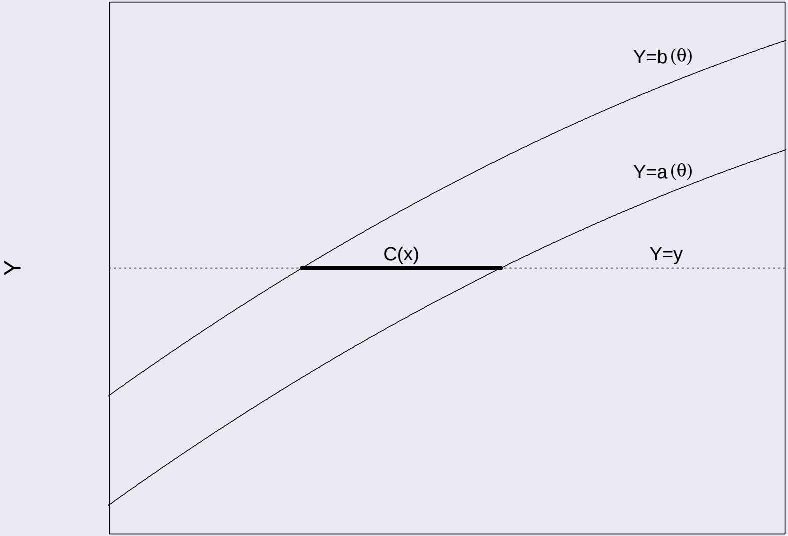
Figure 7.1. A confidence interval obtained by inverting $A(\theta) = [a(\theta), b(\theta)]$



Example 7.7

Suppose that X has the following p.d.f. in a one-parameter exponential family:

$$f_\theta(x) = \exp\{\eta(\theta)Y(x) - \xi(\theta)\}h(x),$$

where θ is real-valued and $\eta(\theta)$ is nondecreasing in θ.
First, we apply Theorem 7.2 with $H_0 : \theta = \theta_0$ and $H_1 : \theta > \theta_0$.
By Theorem 6.2, the acceptance region of the UMP test of size α is

$$A(\theta_0) = \{x : Y(x) \le c(\theta_0)\},$$

where $c(\theta_0) = c$ in Theorem 6.2.
It can be shown that $c(\theta)$ is nondecreasing in θ.
Inverting $A(\theta)$ according to Figure 7.1 with $b(\theta) = c(\theta)$ and $a(\theta)$ ignored, we obtain

$$C(X) = [\underline{\theta}(X), \infty) \quad \text{or} \quad (\underline{\theta}(X), \infty),$$

a one-sided confidence interval for θ with confidence level $1 - \alpha$.
$\underline{\theta}(X)$ is a called a lower confidence bound for θ in §2.4.3.
When the c.d.f. of $Y(X)$ is continuous, $C(X)$ has confidence coefficient $1 - \alpha$.

Example 7.7 (continued)

If $H_0 : \theta = \theta_0$ and $H_1 : \theta < \theta_0$ are considered, then $C(X) = \{\theta : Y(X) \geq c(\theta)\}$ and is of the form

$$(-\infty, \overline{\theta}(X)] \quad \text{or} \quad (-\infty, \overline{\theta}(X)).$$

$\overline{\theta}(X)$ is called an upper confidence bound for θ.

Consider next $H_0 : \theta = \theta_0$ and $H_1 : \theta \neq \theta_0$.
By Theorem 6.4, the acceptance region of the UMPU test of size α is given by $A(\theta_0) = \{x : c_1(\theta_0) \leq Y(x) \leq c_2(\theta_0)\}$, where $c_i(\theta)$ are nondecreasing (exercise).
A confidence interval can be obtained by inverting $A(\theta)$ according to Figure 7.1 with $a(\theta) = c_1(\theta)$ and $b(\theta) = c_2(\theta)$.

Let us consider a specific example in which $X_1, ..., X_n$ are i.i.d. binary random variables with $p = P(X_i = 1)$.
Note that $Y(X) = \sum_{i=1}^n X_i$.
Suppose that we need a lower confidence bound for p so that we consider $H_0 : p = p_0$ and $H_1 : p > p_0$.

Example 7.7 (continued)

If $H_0 : \theta = \theta_0$ and $H_1 : \theta < \theta_0$ are considered, then $C(X) = \{\theta : Y(X) \geq c(\theta)\}$ and is of the form

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Let us consider a specific example in which $X_1, ..., X_n$ are i.i.d. binary random variables with $p = P(X_i = 1)$.
Note that $Y(X) = \sum_{i=1}^n X_i$.
Suppose that we need a lower confidence bound for p so that we consider $H_0 : p = p_0$ and $H_1 : p > p_0$.

Example 7.7 (continued)

From Example 6.2, the acceptance region of a UMP test of size $\alpha \in (0,1)$ is $A(p_0) = \{y : y \le m(p_0)\}$, where $m(p_0)$ is an integer between 0 and n such that

$$\sum_{j=m(p_0)+1}^{n} \binom{n}{j} p_0^j (1-p_0)^{n-j} \le \alpha < \sum_{j=m(p_0)}^{n} \binom{n}{j} p_0^j (1-p_0)^{n-j}.$$

Thus, $m(p)$ is an integer-valued, nondecreasing step-function of p. Define

$$\underline{p} = \inf\{p : m(p) \ge y\} = \inf\left\{p : \sum_{j=y}^{n} \binom{n}{j} p^j (1-p)^{n-j} \ge \alpha\right\}.$$

Then a level $1-\alpha$ confidence interval for p is $(\underline{p}, 1]$ (exercise).
One can compare this confidence interval with the one obtained by applying Theorem 7.1 (exercise).
See also Example 7.16.

Example 7.8

Suppose that X has the following p.d.f. in a multiparameter exponential family:

$$f_{\theta,\varphi}(x) = \exp\{\theta Y(x) + \varphi^\tau U(x) - \zeta(\theta,\varphi)\}$$

By Theorem 6.4, the acceptance region of a UMPU test of size α for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$ or $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is

$$A(\theta_0) = \{(y,u) : y \leq c_2(u,\theta_0)\}$$

or

$$A(\theta_0) = \{(y,u) : c_1(u,\theta_0) \leq y \leq c_2(u,\theta_0)\},$$

where $c_i(u,\theta)$, $i = 1,2$, are nondecreasing functions of θ.
Confidence intervals for θ can then be obtained by inverting $A(\theta)$ according to Figure 7.1 with $b(\theta) = c_2(u,\theta)$ and $a(\theta) = c_1(u,\theta)$ or $a(\theta) \equiv -\infty$, for any observed u.

Example 7.8 (continued)

Consider more specifically the case where X_1 and X_2 are independently distributed as the Poisson distributions $P(\lambda_1)$ and $P(\lambda_2)$, respectively, and we need a lower confidence bound for the ratio $\rho = \lambda_2/\lambda_1$.

From Example 6.11, a UMPU test of size α for testing $H_0 : \rho = \rho_0$ versus $H_1 : \rho > \rho_0$ has the acceptance region

$$A(\rho_0) = \{(y,u) : y \le c(u,\rho_0)\},$$

where $c(u,\rho_0)$ is determined by the conditional distribution of $Y = X_2$ given $U = X_1 + X_2 = u$.

Since the conditional distribution of Y given $U = u$ is the binomial distribution $Bi(\rho/(1+\rho),u)$, we can use the result in Example 7.7, i.e., $c(u,\rho)$ is the same as $m(p)$ in Example 7.7 with $n = u$ and $p = \rho/(1+\rho)$.

Example 7.8 (continued)

Then a level $1-\alpha$ lower confidence bound for p is $\underline{p}$ given by

$$\underline{p} = \inf\{p : m(p) \geq y\} = \inf\left\{ p : \sum_{j=y}^{u} \binom{u}{j} p^j (1-p)^{u-j} \geq \alpha \right\}$$

Since $\rho = p/(1-p)$ is a strictly increasing function of p, a level $1-\alpha$ lower confidence bound for ρ is $\underline{p}/(1-\underline{p})$.

Example 7.9

Consider the normal linear model $X = N_n(Z\beta, \sigma^2 I_n)$ and the problem of constructing a confidence set for $\theta = L\beta$, where L is an $s \times p$ matrix of rank s and all rows of L are in $\mathscr{R}(Z)$.

The LR test of size α for $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ has the acceptance region

$$A(\theta_0) = \{X : W(X, \theta_0) \leq c_\alpha\},$$

Example 7.8 (continued)

Then a level $1-\alpha$ lower confidence bound for p is $\underline{p}$ given by

$$\underline{p} = \inf\{p : m(p) \geq y\} = \inf\left\{p : \sum_{j=y}^{u} \binom{u}{j} p^j (1-p)^{u-j} \geq \alpha\right\}$$

Since $\rho = p/(1-p)$ is a strictly increasing function of p, a level $1-\alpha$ lower confidence bound for ρ is $\underline{p}/(1-\underline{p})$.

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The LR test of size α for $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ has the acceptance region

$$A(\theta_0) = \{X : W(X, \theta_0) \leq c_\alpha\},$$

Example 7.9 (continued)

where c_α is the $(1-\alpha)$th quantile of the F-distribution $F_{s,n-r}$,

$$W(X,\theta) = \frac{[\|X - Z\widehat{\beta}(\theta)\|^2 - \|X - Z\widehat{\beta}\|^2]/s}{\|X - Z\widehat{\beta}\|^2/(n-r)},$$

r is the rank of Z, $r \geq s$, $\widehat{\beta}$ is the LSE of β and, for each fixed θ, $\widehat{\beta}(\theta)$ is a solution of

$$\|X - Z\widehat{\beta}(\theta)\|^2 = \min_{\beta : L\beta = \theta} \|X - Z\beta\|^2.$$

Inverting $A(\theta)$, we obtain the following confidence set for θ with confidence coefficient $1-\alpha$: $C(X) = \{\theta : W(X,\theta) \leq c_\alpha\}$, which forms a closed ellipsoid in $\mathscr{R}^s$.