Stat 710: Mathematical Statistics
Lecture 26

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Lecture 26: UMPU tests in two sample normal problems and linear models

Two-sample problems

The problem of comparing the parameters of two normal distributions arises in the comparison of two treatments, products, and so on.

Suppose that we have two independent samples, $X_{i1},...,X_{in_i}$, $i=1,2$, i.i.d. from $N(\mu_i,\sigma_i^2)$, $i=1,2$, respectively, where $n_i \geq 2$.

The joint p.d.f. of X_{ij}'s is

$$C(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2)\exp\left\{-\sum_{i=1}^{2}\frac{1}{2\sigma_i^2}\sum_{j=1}^{n_i}x_{ij}^2+\sum_{i=1}^{2}\frac{n_i\mu_i}{\sigma_i^2}\bar{x}_i\right\},$$

where $\bar{x}_i$ is the sample mean based on $x_{i1},...,x_{in_i}$ and $C(\cdot)$ is a known function.

Consider first the hypothesis $H_0 : \sigma_2^2/\sigma_1^2 \le \Delta_0$ or $H_0 : \sigma_2^2/\sigma_1^2 = \Delta_0$.
The p.d.f. of X_{ij}'s is in a multiparameter exponential family with

$$\theta = \frac{1}{2\Delta_0\sigma_1^2} - \frac{1}{2\sigma_2^2}, \qquad \varphi = \left(-\frac{1}{2\sigma_1^2}, \frac{n_1\mu_1}{\sigma_1^2}, \frac{n_2\mu_2}{\sigma_2^2}\right),$$

$$Y = \sum_{j=1}^{n_2} X_{2j}^2, \qquad U = \left(\sum_{j=1}^{n_1} X_{1j}^2 + \frac{1}{\Delta_0}\sum_{j=1}^{n_2} X_{2j}^2,\ \bar{X}_1,\ \bar{X}_2\right).$$

To apply Lemma 6.7, consider

$$V = \frac{(n_2-1)S_2^2/\Delta_0}{(n_1-1)S_1^2 + (n_2-1)S_2^2/\Delta_0} = \frac{(Y - n_2U_3)/\Delta_0}{U_1 - n_1U_2 - n_2U_3/\Delta_0},$$

where S_i^2 is the sample variance based on $X_{i1},...,X_{in_i}$ and U_j is the jth component of U.
By Basu's theorem, V and U are independent when $\theta = 0$ ($\sigma_2^2 = \Delta_0\sigma_1^2$).

Two-sample problems

Since V is increasing and linear in Y, the conditions of Lemma 6.7 are satisfied.

Thus, a UMPU test rejects $H_0 : \theta \le 0$ (which is equivalent to $H_0 : \sigma_2^2/\sigma_1^2 \le \Delta_0$) when $V > c_0$, where c_0 satisfies $P(V > c_0) = \alpha$ when $\theta = 0$; and a UMPU test rejects $H_0 : \theta = 0$ (which is equivalent to $H_0 : \sigma_2^2/\sigma_1^2 = \Delta_0$) when $V < c_1$ or $V > c_2$, where c_i's satisfy $P(c_1 < V < c_2) = 1-\alpha$ and $E[VT_*(V)] = \alpha E(V)$ when $\theta = 0$.

Note that

$$V = \frac{(n_2-1)F}{n_1-1+(n_2-1)F} \qquad \text{with} \qquad F = \frac{S_2^2/\Delta_0}{S_1^2}.$$

It follows from Example 1.16 that F has the F-distribution F_{n_2-1,n_1-1} when $\theta = 0$.

Since V is a strictly increasing function of F, a UMPU test rejects $H_0 : \theta \le 0$ when $F > F_{n_2-1,n_1-1,\alpha}$, where $F_{a,b,\alpha}$ is the $(1-\alpha)$th quantile of the F-distribution $F_{a,b}$.

This is the F-test in elementary textbooks.

Two-sample problems

When $\theta = 0$, V has the beta distribution $B((n_2-1)/2,(n_1-1)/2)$ and $E(V) = (n_2-1)/(n_1+n_2-2)$ (Table 1.2).

Then, $E[VT_*(V)] = \alpha E(V)$ when $\theta = 0$ is the same as

$$\frac{(1-\alpha)(n_2-1)}{n_1+n_2-2} = \int_{c_1}^{c_2} v f_{(n_2-1)/2,(n_1-1)/2}(v)dv,$$

where $f_{a,b}$ is the p.d.f. of the beta distribution $B(a,b)$.

Using the fact that

$vf_{(n_2-1)/2,(n_1-1)/2}(v) = (n_1+n_2-2)^{-1}(n_2-1)f_{(n_2+1)/2,(n_1-1)/2}(v)$, we conclude that a UMPU test rejects $H_0 : \theta = 0$ when $V < c_1$ or $V > c_2$, where c_1 and c_2 are determined by

$$1-\alpha = \int_{c_1}^{c_2} v f_{(n_2-1)/2,(n_1-1)/2}(v)dv = \int_{c_1}^{c_2} f_{(n_2+1)/2,(n_1-1)/2}(v)dv.$$

If $n_2 - 1 \approx n_2 + 1$ (i.e., n_2 is large), then this UMPU test can be approximated by the F-test that rejects $H_0 : \theta = 0$ if and only if $F < F_{n_2-1,n_1-1,1-\alpha/2}$ or $F > F_{n_2-1,n_1-1,\alpha/2}$.

Two-sample problems

Consider next the hypothesis $H_0 : \mu_1 \geq \mu_2$ or $H_0 : \mu_1 = \mu_2$.
If $\sigma_1^2 \neq \sigma_2^2$, the problem is the so-called Behrens-Fisher problem and is not accessible by the method introduced in this section.
We now assume that $\sigma_1^2 = \sigma_2^2 = \sigma^2$ but σ^2 is unknown.
The p.d.f. of X_{ij}'s is then

$$C(\mu_1, \mu_2, \sigma^2) \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{2} \sum_{j=1}^{n_i} x_{ij}^2 + \frac{n_1\mu_1}{\sigma^2} \bar{x}_1 + \frac{n_2\mu_2}{\sigma^2} \bar{x}_2 \right\},$$

which is in a multiparameter exponential family with

$$\theta = \frac{\mu_2 - \mu_1}{(n_1^{-1} + n_2^{-1})\sigma^2}, \qquad \varphi = \left(\frac{n_1\mu_1 + n_2\mu_2}{(n_1 + n_2)\sigma^2}, -\frac{1}{2\sigma^2} \right),$$

$$Y = \bar{X}_2 - \bar{X}_1, \qquad U = \left(n_1\bar{X}_1 + n_2\bar{X}_2, \sum_{i=1}^{2} \sum_{j=1}^{n_i} X_{ij}^2 \right).$$

For testing $H_0 : \theta \leq 0$ (i.e., $\mu_1 \geq \mu_2$) versus $H_1 : \theta > 0$, we consider V in Lemma 6.7 to be

$$t(X) = \frac{(\bar{X}_2 - \bar{X}_1)/\sqrt{n_1^{-1} + n_2^{-1}}}{\sqrt{[(n_1-1)S_1^2 + (n_2-1)S_2^2]/(n_1+n_2-2)}}.$$

When $\theta = 0$, $t(X)$ is independent of U (Basu's theorem) and satisfies the conditions in Lemma 6.7(i); the numerator and the denominator of $t(X)$ (after division by σ) are independently distributed as $N(0,1)$ and the chi-square distribution $\chi^2_{n_1+n_2-2}$, respectively.
Hence $t(X)$ has the t-distribution $t_{n_1+n_2-2}$ and a UMPU test rejects H_0 when $t(X) > t_{n_1+n_2-2,\alpha}$ (the $(1-\alpha)$th quantile of $t_{n_1+n_2-2}$).
This is the so-called (one-sided) two-sample t-test.
For testing $H_0 : \theta = 0$ (i.e., $\mu_1 = \mu_2$) versus $H_1 : \theta \neq 0$, it follows from a similar argument used in the derivation of the (two-sided) one-sample t-test that a UMPU test rejects H_0 when $|t(X)| > t_{n_1+n_2-2,\alpha/2}$ (exercise).
This is the (two-sided) two-sample t-test.
The power function of a two-sample t-test is related to a noncentral t-distribution.

Normal linear models

Consider linear model

$$X = (X_1, ..., X_n) \text{ is } N_n(Z\beta, \sigma^2 I_n),$$

where β is a p-vector of unknown parameters, Z is the $n \times p$ matrix whose ith row is the vector Z_i, Z_i's are the values of a p-vector of deterministic covariates, and $\sigma^2 > 0$ is an unknown parameter.
Assume that $n > p$ and the rank of Z is $r \leq p$.
Let $l \in \mathscr{R}(Z)$ (the linear space generated by the rows of Z).
We consider the one-sided hypotheses

$$H_0 : l^\tau \beta \leq \theta_0 \qquad \text{versus} \qquad H_1 : l^\tau \beta > \theta_0$$

or the two-sided hypotheses

$$H_0 : l^\tau \beta = \theta_0 \qquad \text{versus} \qquad H_1 : l^\tau \beta \neq \theta_0.$$

where θ_0 is a fixed constant.
Since $H = Z(Z^\tau Z)^- Z^\tau$ is a projection matrix of rank r, there exists an $n \times n$ orthogonal matrix Γ such that

$$\Gamma = (\ \Gamma_1 \ \ \Gamma_2\ ) \qquad \text{and} \qquad H\Gamma = (\ \Gamma_1 \ \ 0\ ),$$

where Γ_1 is $n \times r$ and Γ_2 is $n \times (n-r)$.

Normal linear models

Let $Y_j = \Gamma_j^\tau X$, $j = 1,2$.
Consider the transformation $(Y_1,Y_2) = \Gamma^\tau X$.
Since $\Gamma^\tau\Gamma = I_n$ and X is $N_n(Z\beta,\sigma^2 I_n)$, (Y_1,Y_2) is $N_n(\Gamma^\tau Z\beta,\sigma^2 I_n)$.
It follows from $H\Gamma_2 = 0$ that

$$E(Y_2) = E(\Gamma_2^\tau X) = \Gamma_2^\tau Z\beta = \Gamma_2^\tau HZ\beta = 0.$$

Let $\eta = \Gamma_1^\tau Z\beta = E(Y_1)$.
Then the p.d.f. of (Y_1,Y_2) is

$$\frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left\{\frac{\eta^\tau Y_1}{\sigma^2} - \frac{\|Y_1\|^2+\|Y_2\|^2}{2\sigma^2} - \frac{\|\eta\|^2}{2\sigma^2}\right\}.$$

Since $l \in \mathscr{R}(Z)$, there exists $\lambda \in \mathscr{R}^n$ such that $l = Z^\tau\lambda$.
Let $\widehat{\beta}$ be the LSE.
Then

$$l^\tau\widehat{\beta} = \lambda^\tau HX = \lambda^\tau\Gamma\Gamma^\tau HX = \lambda^\tau\Gamma_1\Gamma_1^\tau X = \lambda^\tau\Gamma_1 Y_1,$$

and, by the unbiasedness of the LSE,

$$E(l^\tau\widehat{\beta}) = l^\tau\beta = \lambda^\tau\Gamma_1 E(Y_1) = a^\tau\eta, \qquad a = \Gamma_1^\tau\lambda.$$

Normal linear models

Let $\eta = (\eta_1, ..., \eta_r)$ and $a = (a_1, ..., a_r)$.
Without loss of generality, we assume that $a_1 \neq 0$.
Then the p.d.f. of (Y_1, Y_2) is in a multiparameter exponential family with

$$\theta = \frac{a^\tau \eta - \theta_0}{a_1 \sigma^2}, \qquad \varphi = \left(-\frac{1}{2\sigma^2}, \frac{\eta_2}{\sigma^2}, ..., \frac{\eta_r}{\sigma^2}\right), \qquad Y = Y_{11},$$

$$U = \left(\|Y_1\|^2 + \|Y_2\|^2 - \frac{2\theta_0 Y_{11}}{a_1}, Y_{12} - \frac{a_2 Y_{11}}{a_1}, ..., Y_{1r} - \frac{a_r Y_{11}}{a_1}\right),$$

where Y_{1j} is the jth component of Y_1.
By Basu's theorem,

$$t(X) = \frac{\sqrt{n-r}(a^\tau Y_1 - \theta_0)}{\|Y_2\| \, \|a\|}$$

is independent of U when $a^\tau \eta = l^\tau \beta = \theta_0$.

Normal linear models

Note that $\|Y_2\|^2 = SSR$ and $\|a\|^2 = \lambda^\tau \Gamma_1 \Gamma_1^\tau \lambda = \lambda^\tau H \lambda = l^\tau (Z^\tau Z)^- l$.
Hence, by $l^\tau \widehat{\beta} = a^\tau Y_1$,

$$t(X) = \frac{l^\tau \widehat{\beta} - \theta_0}{\sqrt{l^\tau (Z^\tau Z)^- l}\sqrt{SSR/(n-r)}},$$

which has the t-distribution t_{n-r} (Theorem 3.8).
Using the same arguments in deriving the one-sample or two-sample t-test, we obtain that a UMPU test for the one-sided hypotheses rejects H_0 when $t(X) > t_{n-r,\alpha}$, and that a UMPU test for the two-sided hypotheses rejects H_0 when $|t(X)| > t_{n-r,\alpha/2}$.