

Stat 710: Mathematical Statistics

Lecture 25

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Lecture 25: UMPU tests in binomial, Poisson, and one sample normal problems

Example 6.11

A problem arising in many different contexts is the comparison of two treatments.

If the observations are integer-valued, the problem often reduces to testing the equality of two Poisson distributions (e.g., a comparison of the radioactivity of two substances or the car accident rate in two cities) or two binomial distributions (when the observation is the number of successes in a sequence of trials for each treatment).

Consider first the Poisson problem in which X_1 and X_2 are independently distributed as the Poisson distributions $P(\lambda_1)$ and $P(\lambda_2)$, respectively.

The p.d.f. of $X = (X_1, X_2)$ is

$$[e^{-(\lambda_1 + \lambda_2)} / x_1! x_2!] \exp \{ x_2 \log(\lambda_2 / \lambda_1) + (x_1 + x_2) \log \lambda_1 \}$$

w.r.t. the counting measure on $\{(i, j) : i = 0, 1, 2, \dots, j = 0, 1, 2, \dots\}$.

Lecture 25: UMPU tests in binomial, Poisson, and one sample normal problems

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w.r.t. the counting measure on $\{(i, j) : i = 0, 1, 2, \dots, j = 0, 1, 2, \dots\}$.

Example 6.11 (continued)

Let $\theta = \log(\lambda_2/\lambda_1)$.

Then hypotheses such as $\lambda_1 = \lambda_2$ and $\lambda_1 \geq \lambda_2$ are equivalent to $\theta = 0$ and $\theta \leq 0$, respectively.

The p.d.f. of X is in a multiparameter exponential family with $\varphi = \log \lambda_1$, $Y = X_2$, and $U = X_1 + X_2$.

Thus, Theorem 6.4 applies.

To obtain various tests in Theorem 6.4, it is enough to derive the conditional distribution of $Y = X_2$ given $U = X_1 + X_2 = u$.

Using the fact that $X_1 + X_2$ has the Poisson distribution $P(\lambda_1 + \lambda_2)$, one can show that

$$P(Y = y | U = u) = \binom{u}{y} p^y (1-p)^{u-y} I_{\{0,1,\dots,u\}}(y), \quad u = 0, 1, 2, \dots,$$

where $p = \lambda_2/(\lambda_1 + \lambda_2) = e^\theta/(1 + e^\theta)$.

This is the binomial distribution $Bi(p, u)$.

On the boundary set $\bar{\Theta}_{01}$, $\theta = \theta_j$ (a known value) and the distribution $P_{Y|U=u}$ is known.

Example 6.11 (continued)

Consider next the binomial problem in which X_j , $j = 1, 2$, are independently distributed as the binomial distributions $Bi(p_j, n_j)$, $j = 1, 2$, respectively, where n_j 's are known but p_j 's are unknown. The p.d.f. of $X = (X_1, X_2)$ is

$$\binom{n_1}{x_1} \binom{n_2}{x_2} (1-p_1)^{n_1-x_1} (1-p_2)^{n_2-x_2} \exp \left\{ x_2 \log \frac{p_2(1-p_1)}{p_1(1-p_2)} + (x_1 + x_2) \log \frac{p_1}{(1-p_1)} \right\}$$

w.r.t. the counting measure on $\{(i, j) : i = 0, 1, \dots, n_1, j = 0, 1, \dots, n_2\}$.

This p.d.f. is in a multiparameter exponential family with

$$\theta = \log \frac{p_2(1-p_1)}{p_1(1-p_2)}, \quad Y = X_2, \quad \text{and} \quad U = X_1 + X_2.$$

Thus, Theorem 6.4 applies.

Note that hypotheses such as $p_1 = p_2$ and $p_1 \geq p_2$ are equivalent to $\theta = 0$ and $\theta \leq 0$, respectively.

Using the joint distribution of (X_1, X_2) , one can show (exercise) that

$$P(Y = y | U = u) = K_u(\theta) \binom{n_1}{u-y} \binom{n_2}{y} e^{\theta y} I_A(y), \quad u = 0, 1, \dots, n_1 + n_2,$$

Example 6.11 (continued)

where

$$A = \{y : y = 0, 1, \dots, \min\{u, n_2\}, u - y \leq n_1\}$$

and

$$K_u(\theta) = \left[\sum_{y \in A} \binom{n_1}{u-y} \binom{n_2}{y} e^{\theta y} \right]^{-1}.$$

If $\theta = 0$, this distribution reduces to a known distribution: the hypergeometric distribution $HG(u, n_2, n_1)$ (Table 1.1, page 18).

An important application of Theorem 6.4 to problems with continuous distributions in exponential families is the derivation of UMPU tests in normal families.

The results presented here are the basic justifications for tests in elementary textbooks concerning parameters in normal families.

The following lemma is useful especially when X is from a population in a normal family.

Example 6.11 (continued)

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The following lemma is useful especially when X is from a population in a normal family.

Lemma 6.7

Suppose that X has the following p.d.f. w.r.t. a σ -finite measure:

$$f_{\theta, \varphi}(x) = \exp \{ \theta Y(x) + \varphi^{\tau} U(x) - \zeta(\theta, \varphi) \},$$

where θ is a real-valued parameter, φ is a vector-valued parameter, and Y (real-valued) and U (vector-valued) are statistics.

Suppose also that $V(Y, U)$ is a statistic independent of U when $\theta = \theta_j$, where θ_j 's are known values given in the hypotheses in (i)-(iv) of Theorem 6.4.

(i) If $V(y, u)$ is increasing in y for each u , then the UMPU tests in (i)-(iii) of Theorem 6.4 are equivalent to those given by Theorem 6.4 with Y and (Y, U) replaced by V and with $c_i(U)$ and $\gamma_i(U)$ replaced by constants c_i and γ_i , respectively.

(ii) If there are Borel functions $a(u) > 0$ and $b(u)$ such that $V(y, u) = a(u)y + b(u)$, then the UMPU test in Theorem 6.4(iv) is equivalent to that given by Theorem 6.4(iv) with Y and (Y, U) replaced by V and with $c_i(U)$ and $\gamma_i(U)$ replaced by constants c_i and γ_i , respectively.

Proof

(i) Since V is increasing in y , $Y > c_i(u)$ is equivalent to $V > d_i(u)$ for some d_i .

The result follows from the fact that V is independent of U so that d_i 's and γ_i 's do not depend on u when Y is replaced by V .

(ii) Since $V = a(U)Y + b(U)$, the UMPU test in Theorem 6.4(iv) is the same as

$$T_*(V, U) = \begin{cases} 1 & V < c_1(U) \text{ or } V > c_2(U) \\ \gamma_i(U) & V = c_i(U), i = 1, 2, \\ 0 & c_1(U) < V < c_2(U), \end{cases}$$

subject to $E_{\theta_0}[T_*(V, U)|U = u] = \alpha$ and

$$E_{\theta_0} \left[T_*(V, U) \frac{V - b(U)}{a(U)} \middle| U \right] = \alpha E_{\theta_0} \left[\frac{V - b(U)}{a(U)} \middle| U \right]. \quad (1)$$

Under $E_{\theta_0}[T_*(V, U)|U = u] = \alpha$, (1) is the same as

$$E_{\theta_0}[T_*(V, U)V|U] = \alpha E_{\theta_0}(V|U).$$

Since V and U are independent when $\theta = \theta_0$, $c_i(u)$'s and $\gamma_i(u)$'s do not depend on u and, therefore, T_* does not depend on U .

Remarks

- If the conditions of Lemma 6.7 are satisfied, then UMPU tests can be derived by working with the distribution of V instead of $P_{Y|U=u}$.
- In exponential families, a $V(Y, U)$ independent of U can often be found by applying Basu's theorem (Theorem 2.4).
- When we consider normal families, γ_i 's can be chosen to be 0 since the c.d.f. of Y given $U = u$ or the c.d.f. of V is continuous.

One-sample problems

Let X_1, \dots, X_n be i.i.d. from $N(\mu, \sigma^2)$ with unknown $\mu \in \mathcal{R}$ and $\sigma^2 > 0$, where $n \geq 2$.

The joint p.d.f. of $X = (X_1, \dots, X_n)$ is

$$\frac{1}{(2\pi\sigma^2)^{n/2}} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^n x_i^2 + \frac{\mu}{\sigma^2} \sum_{i=1}^n x_i - \frac{n\mu^2}{2\sigma^2} \right\}.$$

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One-sample problems

Consider first hypotheses concerning σ^2 .

The p.d.f. of X is in a multiparameter exponential family with

$\theta = -(2\sigma^2)^{-1}$, $\varphi = n\mu/\sigma^2$, $Y = \sum_{i=1}^n X_i^2$, and $U = \bar{X}$.

By Basu's theorem, $V = (n-1)S^2$ is independent of $U = \bar{X}$ (Example 2.18), where S^2 is the sample variance.

Also,

$$\sum_{i=1}^n X_i^2 = (n-1)S^2 + n\bar{X}^2,$$

i.e., $V = Y - nU^2$.

Hence the conditions of Lemma 6.7 are satisfied.

Since V/σ^2 has the chi-square distribution χ_{n-1}^2 (Example 2.18), values of c_i 's for hypotheses in (i)-(iii) of Theorem 6.4 are related to quantiles of χ_{n-1}^2 .

For testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$ (which is equivalent to testing $H_0 : \sigma^2 = \sigma_0^2$ vs $H_1 : \sigma^2 \neq \sigma_0^2$), $d_i = c_i/\sigma_0^2$, $i = 1, 2$, are determined by

$$\int_{d_1}^{d_2} f_{n-1}(v) dv = 1 - \alpha \quad \text{and} \quad \int_{d_1}^{d_2} v f_{n-1}(v) dv = (n-1)(1 - \alpha),$$

One-sample problems

where f_m is the Lebesgue p.d.f. of the chi-square distribution χ_m^2 . Since $vf_{n-1}(v) = (n-1)f_{n+1}(v)$, d_1 and d_2 are determined by

$$\int_{d_1}^{d_2} f_{n-1}(v) dv = \int_{d_1}^{d_2} f_{n+1}(v) dv = 1 - \alpha.$$

If $n-1 \approx n+1$, then d_1 and d_2 are nearly the $(\alpha/2)$ th and $(1 - \alpha/2)$ th quantiles of χ_{n-1}^2 , respectively, in which case the UMPU test in Theorem 6.4(iv) is the same as the “equal-tailed” chi-square test for H_0 in elementary textbooks.

Consider next hypotheses concerning μ .

The p.d.f. of X has in a multiparameter exponential family with $Y = \bar{X}$, $U = \sum_{i=1}^n (X_i - \mu_0)^2$, $\theta = n(\mu - \mu_0)/\sigma^2$, and $\phi = -(2\sigma^2)^{-1}$.

For testing hypotheses $H_0 : \mu \leq \mu_0$ versus $H_1 : \mu > \mu_0$, we take V to be $t(X) = \sqrt{n}(\bar{X} - \mu_0)/S$.

By Basu's theorem, $t(X)$ is independent of U when $\mu = \mu_0$.

Hence it satisfies the conditions in Lemma 6.7(i).

One-sample problems

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By Basu's theorem, $t(X)$ is independent of U when $\mu = \mu_0$.

Hence it satisfies the conditions in Lemma 6.7(i).

One-sample problems

From Examples 1.16 and 2.18, $t(X)$ has the t-distribution t_{n-1} when $\mu = \mu_0$.

Thus, $c(U)$ in Theorem 6.4(i) is the $(1 - \alpha)$ th quantile of t_{n-1} .

For the two-sided hypotheses $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$, the statistic $V = (\bar{X} - \mu_0)/\sqrt{U}$ satisfies the conditions in Lemma 6.7(ii) and has a distribution symmetric about 0 when $\mu = \mu_0$.

Then the UMPU test in Theorem 6.4(iv) rejects H_0 when $|V| > d$, where d satisfies $P(|V| > d) = \alpha$ when $\mu = \mu_0$.

Since

$$t(X) = \sqrt{(n-1)n}V(X)/\sqrt{1 - n[V(X)]^2},$$

the UMPU test rejects H_0 if and only if $|t(X)| > t_{n-1, \alpha/2}$, where $t_{n-1, \alpha}$ is the $(1 - \alpha)$ th quantile of the t-distribution t_{n-1} .

The UMPU tests derived here are the so-called one-sample t-tests in elementary textbooks.

The power function of a one-sample t-test is related to the noncentral t-distribution introduced in §1.3.1 (see Exercise 36).