Stat 710: Mathematical Statistics
Lecture 24

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Consider the following hypotheses:

$$H_0 : \theta \in \Theta_0 \qquad \text{versus} \qquad H_1 : \theta \in \Theta_1,$$

where $\theta = \theta(P)$ is a functional from $\mathscr{P}$ onto Θ and Θ_0 and Θ_1 are two disjoint Borel sets with $\Theta_0 \cup \Theta_1 = \Theta$. ($\mathscr{P}_j = \{P : \theta \in \Theta_j\}$, $j = 0, 1$.)
For instance, $X_1, ..., X_n$ are i.i.d. from F but we are interested in testing $H_0 : \theta \le 0$ versus $H_1 : \theta > 0$, where $\theta = EX_1$.

Definition 6.4 (Similarity)

Consider the hypotheses $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$.
Let α be a given level of significance and let $\bar{\Theta}_{01}$ be the common boundary of Θ_0 and Θ_1, i.e., the set of points θ that are points or limit points of both Θ_0 and Θ_1.
A test T is *similar* on $\bar{\Theta}_{01}$ if and only if $\beta_T(P) = \alpha$ for all $\theta \in \bar{\Theta}_{01}$.

Remark

It is more convenient to work with similarity than to work with unbiasedness for testing $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$.

Lecture 24: UMPU tests in exponential families

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Continuity of the power function

For a given test T, the power function $\beta_T(P)$ is said to be continuous in θ if and only if for any $\{\theta_j : j = 0, 1, 2, ...\} \subset \Theta$, $\theta_j \to \theta_0$ implies $\beta_T(P_j) \to \beta_T(P_0)$, where $P_j \in \mathscr{P}$ satisfying $\theta(P_j) = \theta_j$, $j = 0, 1, ...$
If β_T is a function of θ, then this continuity property is simply the continuity of $\beta_T(\theta)$.

Lemma 6.5

Consider hypotheses $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$.
Suppose that, for every T, $\beta_T(P)$ is continuous in θ.
If T_* is uniformly most powerful among all similar tests and has size α, then T_* is a UMPU test.

Proof

Under the continuity assumption on β_T, the class of similar tests contains the class of unbiased tests.
Since T_* is uniformly at least as powerful as the test $T \equiv \alpha$, T_* is unbiased.
Hence, T_* is a UMPU test.

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Neyman structure

Let $U(X)$ be a sufficient statistic for $P \in \bar{\mathscr{P}}$ and let $\bar{\mathscr{P}}_U$ be the family of distributions of U as P ranges over $\bar{\mathscr{P}}$.
A test is said to have *Neyman structure* w.r.t. U if

$$E[T(X)|U] = \alpha \qquad \text{a.s. } \bar{\mathscr{P}}_U,$$

Clearly, if T has Neyman structure, then

$$E[T(X)] = E\{E[T(X)|U]\} = \alpha \qquad P \in \bar{\mathscr{P}},$$

i.e., T is similar on $\bar{\Theta}_{01}$.
If all tests similar on $\bar{\Theta}_{01}$ have Neyman structure w.r.t. U, then working with tests having Neyman structure is the same as working with tests similar on $\bar{\Theta}_{01}$.

Lemma 6.6

Let $U(X)$ be a sufficient statistic for $P \in \bar{\mathscr{P}}$.
A necessary and sufficient condition for all tests similar on $\bar{\Theta}_{01}$ to have Neyman structure w.r.t. U is that U is boundedly complete for $P \in \bar{\mathscr{P}}$.

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Lemma 6.6

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A necessary and sufficient condition for all tests similar on $\bar{\Theta}_{01}$ to have Neyman structure w.r.t. U is that U is boundedly complete for $P \in \bar{\mathscr{P}}$.

Proof

(i) Suppose first that U is boundedly complete for $P \in \bar{\mathscr{P}}$.
Let $T(X)$ be a test similar on $\bar{\Theta}_{01}$.
Then $E[T(X) - \alpha] = 0$ for all $P \in \bar{\mathscr{P}}$.
From the boundedness of $T(X)$, $E[T(X)|U]$ is bounded.
Since $E\{E[T(X)|U] - \alpha\} = E[T(X) - \alpha] = 0$ for all $P \in \bar{\mathscr{P}}$ and U is boundedly complete, $E[T(X)|U] = \alpha$ a.s. $\bar{\mathscr{P}}_U$, i.e., T has Neyman structure.
(ii) Suppose now that all tests similar on $\bar{\Theta}_{01}$ have Neyman structure w.r.t. U.
Suppose also that U is not boundedly complete for $P \in \bar{\mathscr{P}}$.
Then there is a function h such that $|h(u)| \leq C$, $E[h(U)] = 0$ for all $P \in \bar{\mathscr{P}}$, and $h(U) \neq 0$ with positive probability for some $P \in \bar{\mathscr{P}}$.
Let $T(X) = \alpha + ch(U)$, where $c = \min\{\alpha, 1 - \alpha\}/C$.
Then T is a test similar on $\bar{\Theta}_{01}$ but T does not have Neyman structure w.r.t. U (because $h(U) \neq 0$).
Thus, U must be boundedly complete for $P \in \bar{\mathscr{P}}$.
This proves the result.

Theorem 6.4 (UMPU tests in multiparameter exponential families)

Suppose that X has the following p.d.f. w.r.t. a σ-finite measure:

$$f_{\theta,\varphi}(x) = \exp\{\theta Y(x) + \varphi^\tau U(x) - \zeta(\theta,\varphi)\},$$

where θ is a real-valued parameter, φ is a vector-valued parameter, and Y (real-valued) and U (vector-valued) are statistics.

(i) For testing $H_0 : \theta \le \theta_0$ versus $H_1 : \theta > \theta_0$, a UMPU test of size α is

$$T_*(Y,U) = \begin{cases} 1 & Y > c(U) \\ \gamma(U) & Y = c(U) \\ 0 & Y < c(U), \end{cases}$$

where $c(u)$ and $\gamma(u)$ are Borel functions determined by

$$E_{\theta_0}[T_*(Y,U)|U=u] = \alpha \text{ for every } u$$

and E_{θ_0} is the expectation w.r.t. $f_{\theta_0,\varphi}$.

(ii) For testing $H_0 : \theta \le \theta_1$ or $\theta \ge \theta_2$ versus $H_1 : \theta_1 < \theta < \theta_2$, a UMPU test of size α is

$$T_*(Y,U) = \begin{cases} 1 & c_1(U) < Y < c_2(U) \\ \gamma_i(U) & Y = c_i(U),\ i = 1,2, \\ 0 & Y < c_1(U) \text{ or } Y > c_2(U), \end{cases}$$

Theorem 6.4 (continued)

where $c_i(u)$'s and $\gamma_i(u)$'s are Borel functions determined by

$$E_{\theta_1}[T_*(Y,U)|U=u] = E_{\theta_2}[T_*(Y,U)|U=u] = \alpha \text{ for every } u.$$

(iii) For testing $H_0 : \theta_1 \le \theta \le \theta_2$ versus $H_1 : \theta < \theta_1$ or $\theta > \theta_2$, a UMPU test of size α is

$$T_*(Y,U) = \begin{cases} 1 & Y < c_1(U) \text{ or } Y > c_2(U) \\ \gamma_i(U) & Y = c_i(U),\ i = 1,2, \\ 0 & c_1(U) < Y < c_2(U), \end{cases}$$

where $c_i(u)$'s and $\gamma_i(u)$'s are Borel functions determined by

$$E_{\theta_1}[T_*(Y,U)|U=u] = E_{\theta_2}[T_*(Y,U)|U=u] = \alpha \text{ for every } u.$$

(iv) For testing $H_0 : \theta = \theta_0$ versus $H_1 :\ \theta \neq \theta_0$, a UMPU test of size α is given by $T_*(Y,U)$ in (iii), where $c_i(u)$'s and $\gamma_i(u)$'s are Borel functions determined by

$$E_{\theta_0}[T_*(Y,U)|U=u] = \alpha \text{ for every } u$$

and

$$E_{\theta_0}[T_*(Y,U)Y|U=u] = \alpha E_{\theta_0}(Y|U=u) \text{ for every } u.$$

Proof

By sufficiency, we only need to consider tests that are functions of (Y,U).

It follows from Theorem 2.1(i) that the p.d.f. of (Y,U) (w.r.t. a σ-finite measure) is in a natural exponential family of the form $\exp\{\theta y+\varphi^\tau u-\zeta(\theta,\varphi)\}$ and, given $U=u$, the p.d.f. of the conditional distribution of Y (w.r.t. a σ-finite measure ν_u) is in a natural exponential family of the form $\exp\{\theta y-\zeta_u(\theta)\}$.

Hypotheses in (i)-(iv) are of the form $H_0:\theta\in\Theta_0$ vs $H_1:\theta\in\Theta_1$ with $\bar{\Theta}_{01}=\{(\theta,\varphi):\theta=\theta_0\}$ or = $\{(\theta,\varphi):\theta=\theta_i, i=1,2\}$.

In case (i) or (iv), U is sufficient and complete for $P\in\bar{\mathscr{P}}$ and, hence, Lemma 6.6 applies.

In case (ii) or (iii), applying Lemma 6.6 to each $\{(\theta,\varphi):\theta=\theta_i\}$ also shows that working with tests having Neyman structure is the same as working with tests similar on $\bar{\Theta}_{01}$.

By Theorem 2.1, the power functions of all tests are continuous and, hence, Lemma 6.5 applies.

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By Theorem 2.1, the power functions of all tests are continuous and, hence, Lemma 6.5 applies.

Proof (continued)

Thus, for (i), it suffices to show T_* is UMP among all tests T satisfying

$$E_{\theta_0}[T(Y,U)|U=u]=\alpha \text{ for every } u \tag{1}$$

and for part (ii) or (iii)), it suffices show T_* is UMP among all tests T satisfying

$$E_{\theta_1}[T(Y,U)|U=u]=E_{\theta_2}[T(Y,U)|U=u]=\alpha \text{ for every } u.$$

For (iv), any unbiased T should satisfy (1) and

$$\frac{\partial}{\partial\theta}E_{\theta,\varphi}[T(Y,U)]=0, \qquad \theta\in\bar{\Theta}_{01}. \tag{2}$$

One can show (exercise) that (2) is equivalent to

$$E_{\theta,\varphi}[T(Y,U)Y-\alpha Y]=0, \qquad \theta\in\bar{\Theta}_{01}. \tag{3}$$

Using the argument in the proof of Lemma 6.6, one can show (exercise) that (3) is equivalent to

$$E_{\theta_0}[T(Y,U)Y|U=u]=\alpha E_{\theta_0}(Y|U=u) \text{ for every } u. \tag{4}$$

Hence, for (iv), it suffices to show T_* is UMP among all tests T satisfying (1) and (4).

Proof (continued)

Note that the power function of any test $T(Y,U)$ is

$$\beta_T(\theta,\varphi) = \int \left[\int T(y,u) dP_{Y|U=u}(y) \right] dP_U(u).$$

Thus, it suffices to show that for every fixed u and $\theta \in \Theta_1$, T_* maximizes

$$\int T(y,u) dP_{Y|U=u}(y)$$

over all T subject to the given side conditions.
Since $P_{Y|U=u}$ is in a one-parameter exponential family, the results in (i) and (ii) follow from Corollary 6.1 and Theorem 6.3, respectively.
The result in (iii) follows from Theorem 6.3(ii) by considering $1 - T_*$.

To prove the result in (iv), it suffices to show that if Y has the p.d.f. given by $\exp\{\theta y - \zeta_u(\theta)\}$ and if u is treated as a constant in (1) and (4), T_* in (iii) with a fixed u is UMP subject to conditions (1) and (4).
We now omit u in the following proof for (iv), which is very similar to the proof of Theorem 6.3.

Proof (continued)

Note that the power function of any test $T(Y,U)$ is

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We now omit u in the following proof for (iv), which is very similar to the proof of Theorem 6.3.

Proof (continued)

First, $(\alpha, \alpha E_{\theta_0}(Y))$ is an interior point of the set of points $(E_{\theta_0}[T(Y)], E_{\theta_0}[T(Y)Y])$ as T ranges over all tests of the form $T(Y)$.
By Lemma 6.2 and Proposition 6.1, for testing $\theta = \theta_0$ versus $\theta = \theta_1$, the UMP test is equal to 1 when

$$(k_1 + k_2 y)e^{\theta_0 y} < C(\theta_0, \theta_1)e^{\theta_1 y},$$

where k_i's and $C(\theta_0, \theta_1)$ are constants.
This inequality is equivalent to

$$a_1 + a_2 y < e^{by}$$

for some constants a_1, a_2, and b.
This region is either one-sided or the outside of an interval.
By Theorem 6.2(ii), a one-sided test has a strictly monotone power function and therefore cannot satisfy (4).
Thus, this test must have the form of T_* in (iii).
Since T_* in (iii) does not depend on θ_1, by Lemma 6.1, it is UMP over all tests satisfying (1) and (4); in particular, the test $\equiv \alpha$.
Thus, T_* is UMPU.

Proof (continued)

Finally, it can be shown that all the c- and γ-functions in (i)-(iv) are Borel functions of u (see Lehmann (1986, p. 149)).