Stat 710: Mathematical Statistics
Lecture 23

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Lecture 23: UMP tests for two-sided hypotheses and unbiased tests

Proposition 6.1 (Generalized Neyman-Pearson lemma)

Let $f_1,...,f_{m+1}$ be Borel functions on $\mathscr{R}^p$ integrable w.r.t. a σ-finite ν. For given constants $t_1,...,t_m$, let $\mathscr{T}$ be the class of Borel functions ϕ (from $\mathscr{R}^p$ to $[0,1]$) satisfying

$$\int \phi f_i d\nu \le t_i, \quad i = 1,...,m, \tag{1}$$

and $\mathscr{T}_0$ be the set of ϕ's in $\mathscr{T}$ satisfying (1) with all inequalities replaced by equalities. If there are constants $c_1,...,c_m$ such that

$$\phi_*(x) = \begin{cases} 1 & f_{m+1}(x) > c_1 f_1(x) + \cdots + c_m f_m(x) \\ 0 & f_{m+1}(x) < c_1 f_1(x) + \cdots + c_m f_m(x) \end{cases}$$

is a member of $\mathscr{T}_0$, then ϕ_* maximizes $\int \phi f_{m+1} d\nu$ over $\phi \in \mathscr{T}_0$.
If $c_i \ge 0$ for all i, then ϕ_* maximizes $\int \phi f_{m+1} d\nu$ over $\phi \in \mathscr{T}$.

The proof is left as an exercise.
The result is useful for finding optimal tests for two sided hypotheses.

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The existence of constants c_i's in ϕ_* is considered in the following lemma whose proof can be found in Lehmann (1986, pp. 97-99).

Lemma 6.2

Let $f_1,...,f_m$ and ν be given by Proposition 6.1.
Then the set $M = \left\{\left(\int \phi f_1 d\nu, ..., \int \phi f_m d\nu\right) : \ \phi \text{ is from } \mathscr{R}^p \text{ to } [0,1]\right\}$ is convex and closed.
If $(t_1,...,t_m)$ is an interior point of M, then there exist constants $c_1,...,c_m$ such that the function ϕ_* defined in Proposition 6.1 is in $\mathscr{T}_0$.

Two-sided hypotheses

The following hypotheses are called two-sided hypotheses:

$$H_0: \ \theta \le \theta_1 \text{ or } \theta \ge \theta_2 \quad \text{versus} \quad H_1: \ \theta_1 < \theta < \theta_2, \qquad (2)$$

$$H_0: \ \theta_1 \le \theta \le \theta_2 \quad \text{versus} \quad H_1: \ \theta < \theta_1 \text{ or } \theta > \theta_2, \qquad (3)$$

$$H_0: \ \theta = \theta_0 \quad \text{versus} \quad H_1: \ \theta \ne \theta_0, \qquad (4)$$

where θ_0, θ_1, and θ_2 are given constants and $\theta_1 < \theta_2$.

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where θ_0, θ_1, and θ_2 are given constants and $\theta_1 < \theta_2$.

Theorem 6.3 (UMP tests for two-sided hypotheses)

Suppose that X has a p.d.f. in a one-parameter exponential family, i.e., the p.d.f. is

$$f_\theta(x) = \exp\{\eta(\theta)Y(x) - \xi(\theta)\}h(x)$$

w.r.t. a σ-finite measure, where η is a strictly increasing function of θ.
(i) For testing hypotheses (2), a UMP test of size α is

$$T_*(X) = \begin{cases} 1 & c_1 < Y(X) < c_2 \\ \gamma_i & Y(X) = c_i,\ i = 1,2 \\ 0 & Y(X) < c_1 \text{ or } Y(X) > c_2, \end{cases} \qquad (5)$$

where c_i's and γ_i's are determined by

$$\beta_{T_*}(\theta_1) = \beta_{T_*}(\theta_2) = \alpha. \qquad (6)$$

(ii) T_* minimizes $\beta_T(\theta)$ over all $\theta < \theta_1$, $\theta > \theta_2$, and T satisfying (6).
(iii) If T_* and T_{**} are two tests satisfying (5) and $\beta_{T_*}(\theta_1) = \beta_{T_{**}}(\theta_1)$ and if the region $\{T_{**} = 1\}$ is to the right of $\{T_* = 1\}$, then $\beta_{T_*}(\theta) < \beta_{T_{**}}(\theta)$ for $\theta > \theta_1$ and $\beta_{T_*}(\theta) > \beta_{T_{**}}(\theta)$ for $\theta < \theta_1$.
If both T_* and T_{**} satisfy (5) and (6), then $T_* = T_{**}$ a.s. $\mathscr{P}$.

Proof

(i) Since Y is sufficient for θ, we only need to consider tests of the form $T(Y)$.

By Theorem 2.1, the distribution of Y has a p.d.f.

$$g_\theta(y) = \exp\{\eta(\theta)y - \xi(\theta)\} \tag{7}$$

Let $\theta_1 < \theta_3 < \theta_2$.

Consider the problem of testing $\theta = \theta_1$ or $\theta = \theta_2$ versus $\theta = \theta_3$.

(α, α) is an interior point of the set of all points $(\beta_T(\theta_1), \beta_T(\theta_2))$ as T ranges over all tests of the form $T(Y)$.

By (7) and Lemma 6.2, there are constants $\tilde{c}_1$ and $\tilde{c}_2$ such that

$$T_*(Y) = \begin{cases} 1 & a_1 e^{b_1 Y} + a_2 e^{b_2 Y} < 1 \\ 0 & a_1 e^{b_1 Y} + a_2 e^{b_2 Y} > 1 \end{cases}$$

satisfies (6), where $a_i = \tilde{c}_i e^{\xi(\theta_3) - \xi(\theta_i)}$ and $b_i = \eta(\theta_i) - \eta(\theta_3)$, $i = 1, 2$.

Clearly a_i's cannot both be ≤ 0.

Proof (continued)

If one of the a_i's is ≤ 0 and the other is > 0, then $a_1 e^{b_1 Y} + a_2 e^{b_2 Y}$ is strictly monotone (since $b_1 < 0 < b_2$) and

$$T_*(\text{ or } 1 - T_*) = \begin{cases} 1 & Y(X) > c \\ \gamma & Y(X) = c \\ 0 & Y(X) < c, \end{cases}$$

which has a strictly monotone power function (Theorem 6.2) and, therefore, cannot satisfy $\beta_{T_*}(\theta_1) = \beta_{T_*}(\theta_2) = \alpha$.

Thus, both a_i's are positive.

The function $a_1 e^{b_1 Y} + a_2 e^{b_2 Y}$ is convex (since $b_1 < 0 < b_2$).

$a_1 e^{b_1 Y} + a_2 e^{b_2 Y} < 1$ is equivalent to $c_1 < Y < c_2$ for some c_1 and c_2.

Then, T_* is of the form (5) and it follows from Proposition 6.1 that T_* is UMP for testing $\theta = \theta_1$ or $\theta = \theta_2$ versus $\theta = \theta_3$.

Since T_* does not depend on θ_3, it follows from Lemma 6.1 that T_* is UMP for testing $\theta = \theta_1$ or $\theta = \theta_2$ versus H_1.

To show that T_* is a UMP test of size α for testing H_0 versus H_1, it remains to show that $\beta_{T_*}(\theta) \leq \alpha$ for $\theta \leq \theta_1$ or $\theta \geq \theta_2$, which follows from part (ii) of the theorem by comparing T_* with the test $T(Y) \equiv \alpha$.

Example 6.10

Let $X_1, ..., X_n$ be i.i.d. from $N(\theta, 1)$.
By Theorem 6.3, a UMP test for testing (2) is $T_*(X) = I_{(c_1,c_2)}(\bar{X})$, where c_i's are determined by

$$\Phi\left(\sqrt{n}(c_2 - \theta_1)\right) - \Phi\left(\sqrt{n}(c_1 - \theta_1)\right) = \alpha$$

and

$$\Phi\left(\sqrt{n}(c_2 - \theta_2)\right) - \Phi\left(\sqrt{n}(c_1 - \theta_2)\right) = \alpha.$$

When the distribution of X is not from a one-parameter exponential family, UMP tests for hypotheses (2) exist in some cases (see Exercises 17 and 26).

Unfortunately, a UMP test does not exist in general for testing hypotheses (3) or (4) (Exercises 28 and 29).

A key reason for this phenomenon is that UMP tests for testing one-sided hypotheses do not have level α for testing (2);
but they are of level α for testing (3) or (4) and there does not exist a single test more powerful than all tests that are UMP for testing one-sided hypotheses.

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Unbiased tests

When a UMP test does not exist, we may use the same approach used in estimation problems, i.e., imposing a reasonable restriction on the tests to be considered and finding optimal tests within the class of tests under the restriction.
Two such types of restrictions in estimation problems are unbiasedness and invariance.
A UMP test T of size α has the property that

$$\beta_T(P) \le \alpha, \quad P \in \mathscr{P}_0 \qquad \text{and} \qquad \beta_T(P) \ge \alpha, \quad P \in \mathscr{P}_1, \tag{8}$$

since T is at least as good as the silly test $T \equiv \alpha$.
This leads to the following definition.

Definition 6.3

Let α be a given level of significance.
A test T for $H_0 : P \in \mathscr{P}_0$ versus $H_1 : P \in \mathscr{P}_1$ is said to be unbiased of level α if and only if (8) holds.
A test of size α is called a *uniformly most powerful unbiased* (UMPU) test if and only if it is UMP within the class of unbiased tests of level α.

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Discussion

Since a UMP test is UMPU, the discussion of unbiasedness of tests is useful only when a UMP test does not exist.
In a large class of problems for which a UMP test does not exist, there do exist UMPU tests.

Suppose that U is a sufficient statistic for $P \in \mathscr{P}$.
Then, similar to the search for a UMP test, we need to consider functions of U only in order to find a UMPU test, since, for any unbiased test $T(X)$, $E(T|U)$ is unbiased and has the same power function as T.