

# Stat 709: Mathematical Statistics

## Lecture 37

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## Lecture 37: Robustness of LSE's

Consider model

$$X = Z\beta + \varepsilon. \quad (1)$$

under assumption A3 ( $E(\varepsilon) = 0$  and  $\text{Var}(\varepsilon)$  is an unknown matrix).

An interesting question is under what conditions on  $\text{Var}(\varepsilon)$  is the LSE of  $I^r\beta$  with  $I \in \mathcal{R}(Z)$  still the BLUE.

If  $I^r\hat{\beta}$  is still the BLUE, then we say that  $I^r\hat{\beta}$ , considered as a BLUE, is *robust* against violation of assumption A2.

A procedure having certain properties under an assumption is said to be robust against violation of the assumption iff the procedure still has the same properties when the assumption is (slightly) violated.

For example, the LSE of  $I^r\beta$  with  $I \in \mathcal{R}(Z)$ , as an unbiased estimator, is robust against violation of assumption A1 or A2, since the LSE is unbiased as long as  $E(\varepsilon) = 0$ , which can be always assumed.

On the other hand, the LSE as a UMVUE may not be robust against violation of assumption A1.

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On the other hand, the LSE as a UMVUE may not be robust against violation of assumption A1.

## Theorem 3.10

Consider model (1) with assumption A3.

The following are equivalent.

- (a)  $l^T \hat{\beta}$  is the BLUE of  $l^T \beta$  for any  $l \in \mathcal{R}(Z)$ .
- (b)  $E(l^T \hat{\beta} \eta^T X) = 0$  for any  $l \in \mathcal{R}(Z)$  and any  $\eta$  such that  $E(\eta^T X) = 0$ .
- (c)  $Z^T \text{Var}(\varepsilon) U = 0$ , where  $U$  is a matrix such that  $Z^T U = 0$  and  $\mathcal{R}(U^T) + \mathcal{R}(Z^T) = \mathcal{R}^n$ .
- (d)  $\text{Var}(\varepsilon) = Z \Lambda_1 Z^T + U \Lambda_2 U^T$  for some  $\Lambda_1$  and  $\Lambda_2$ .
- (e) The matrix  $Z(Z^T Z)^- Z^T \text{Var}(\varepsilon)$  is symmetric.

## Proof

We first show that (a) and (b) are equivalent, which is an analogue of Theorem 3.2(i).

Suppose that (b) holds.

Let  $l \in \mathcal{R}(Z)$ .

If  $c^T X$  is unbiased for  $l^T \beta$ , then  $E(\eta^T X) = 0$  with  $\eta = c - Z(Z^T Z)^- l$ .

Hence, (b) implies (a) because

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## Proof (continued)

$$\begin{aligned}\text{Var}(\mathbf{c}^\tau \mathbf{X}) &= \text{Var}(\mathbf{c}^\tau \mathbf{X} - I^\tau \widehat{\beta} + I^\tau \widehat{\beta}) \\ &= \text{Var}(\mathbf{c}^\tau \mathbf{X} - I^\tau (\mathbf{Z}^\tau \mathbf{Z})^{-1} \mathbf{Z}^\tau \mathbf{X} + I^\tau \widehat{\beta}) \\ &= \text{Var}(\eta^\tau \mathbf{X} + I^\tau \widehat{\beta}) \\ &= \text{Var}(\eta^\tau \mathbf{X}) + \text{Var}(I^\tau \widehat{\beta}) + 2 \text{Cov}(\eta^\tau \mathbf{X}, I^\tau \widehat{\beta}) \\ &= \text{Var}(\eta^\tau \mathbf{X}) + \text{Var}(I^\tau \widehat{\beta}) + 2E(I^\tau \widehat{\beta} \eta^\tau \mathbf{X}) \\ &= \text{Var}(\eta^\tau \mathbf{X}) + \text{Var}(I^\tau \widehat{\beta}) \\ &\geq \text{Var}(I^\tau \widehat{\beta}).\end{aligned}$$

Suppose now that there are  $l \in \mathcal{R}(\mathbf{Z})$  and  $\eta$  such that  $E(\eta^\tau \mathbf{X}) = 0$  but  $\delta = E(I^\tau \widehat{\beta} \eta^\tau \mathbf{X}) \neq 0$ .

Let  $\mathbf{c}_t = t\eta + \mathbf{Z}(\mathbf{Z}^\tau \mathbf{Z})^{-1}l$ .

From the previous proof,

$$\text{Var}(\mathbf{c}_t^\tau \mathbf{X}) = t^2 \text{Var}(\eta^\tau \mathbf{X}) + \text{Var}(I^\tau \widehat{\beta}) + 2\delta t.$$

As long as  $\delta \neq 0$ , there exists a  $t$  such that  $\text{Var}(\mathbf{c}_t^\tau \mathbf{X}) < \text{Var}(I^\tau \widehat{\beta})$ .

## Proof (continued)

This shows that  $I^T \hat{\beta}$  cannot be a BLUE and, therefore, (a) implies (b).

Next, we show that (b) implies (c).

Suppose that (b) holds.

Since  $l \in \mathcal{R}(Z)$ ,  $l = Z^T \gamma$  for some  $\gamma$ .

Let  $\eta \in \mathcal{R}(U^T)$ .

Then  $E(\eta^T X) = \eta^T Z \beta = 0$  and, hence,

$$0 = E(l^T \hat{\beta} \eta^T X) = E[\gamma^T Z (Z^T Z)^{-1} Z^T X X^T \eta] = \gamma^T Z (Z^T Z)^{-1} Z^T \text{Var}(\varepsilon) \eta.$$

Since this equality holds for all  $l \in \mathcal{R}(Z)$ , it holds for all  $\gamma$ .

Thus,

$$Z (Z^T Z)^{-1} Z^T \text{Var}(\varepsilon) U = 0,$$

which implies

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since  $Z^T Z (Z^T Z)^{-1} Z^T = Z^T$ .

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## Proof (continued)

Next, we show that (c) implies (d).

We need to use the following facts from the theory of linear algebra: there exists a nonsingular matrix  $C$  such that  $\text{Var}(\varepsilon) = CC^T$  and  $C = ZC_1 + UC_2$  for some matrices  $C_j$  (since  $\mathcal{R}(U^T) + \mathcal{R}(Z^T) = \mathcal{R}^n$ ).

Let  $\Lambda_1 = C_1 C_1^T$ ,  $\Lambda_2 = C_2 C_2^T$ , and  $\Lambda_3 = C_1 C_2^T$ .

Then

$$\text{Var}(\varepsilon) = Z\Lambda_1 Z^T + U\Lambda_2 U^T + Z\Lambda_3 U^T + U\Lambda_3^T Z^T \quad (2)$$

and  $Z^T \text{Var}(\varepsilon) U = Z^T Z\Lambda_3 U^T U$ , which is 0 if (c) holds.

Hence, (c) implies

$$0 = Z(Z^T Z)^{-1} Z^T Z\Lambda_3 U^T U (U^T U)^{-1} U^T = Z\Lambda_3 U^T,$$

which with (2) implies (d).

We now show that (d) implies (e).

If (d) holds, then  $Z(Z^T Z)^{-1} Z^T \text{Var}(\varepsilon) = Z\Lambda_1 Z^T$ , which is symmetric.

Hence (d) implies (e).

To complete the proof, we need to show that (e) implies (b), which is left as an exercise.

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As a corollary of this theorem, the following result shows when the UMVUE's in model (1) with assumption A1 are robust against the violation of  $\text{Var}(\varepsilon) = \sigma^2 I_n$ .

### Corollary 3.3

Consider model (1) with a full rank  $Z$ ,  $\varepsilon = N_n(0, \Sigma)$ , and an unknown positive definite matrix  $\Sigma$ .

Then  $I^r \hat{\beta}$  is a UMVUE of  $I^r \beta$  for any  $I \in \mathcal{R}^p$  iff one of (b)-(e) in Theorem 3.10 holds.

### Example 3.16

Consider model (1) with  $\beta$  replaced by a random vector  $\vec{\beta}$  that is independent of  $\varepsilon$ .

Such a model is called a linear model with random coefficients.

Suppose that  $\text{Var}(\varepsilon) = \sigma^2 I_n$  and  $E(\vec{\beta}) = \beta$ .

Then

$$X = Z\beta + Z(\vec{\beta} - \beta) + \varepsilon = Z\beta + e, \quad (3)$$

where  $e = Z(\vec{\beta} - \beta) + \varepsilon$  satisfies  $E(e) = 0$  and

$$\text{Var}(e) = Z \text{Var}(\vec{\beta}) Z^T + \sigma^2 I_n$$

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Since

$$Z(Z^T Z)^{-1} Z^T \text{Var}(\mathbf{e}) = Z \text{Var}(\vec{\beta}) Z^T + \sigma^2 Z(Z^T Z)^{-1} Z^T$$

is symmetric, by Theorem 3.10, the LSE  $I^T \hat{\beta}$  under model (3) is the BLUE for any  $I^T \beta$ ,  $I \in \mathcal{R}(Z)$ .

If  $Z$  is of full rank and  $\varepsilon$  is normal, then, by Corollary 3.3,  $I^T \hat{\beta}$  is the UMVUE of  $I^T \beta$  for any  $I \in \mathcal{R}^p$ .

### Example 3.17 (Random effects models)

Suppose that

$$X_{ij} = \mu + A_i + e_{ij}, \quad j = 1, \dots, n_i, i = 1, \dots, m, \quad (4)$$

where  $\mu \in \mathcal{R}$  is an unknown parameter,  $A_i$ 's are i.i.d. random variables having mean 0 and variance  $\sigma_a^2$ ,  $e_{ij}$ 's are i.i.d. random errors with mean 0 and variance  $\sigma^2$ , and  $A_i$ 's and  $e_{ij}$ 's are independent.

Model (4) is called a one-way *random effects* model and  $A_i$ 's are unobserved random effects.

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Model (4) is a special case of model (1) with  $\varepsilon_{ij} = A_i + e_{ij}$  and

$$\text{Var}(\varepsilon) = \sigma_a^2 \Sigma + \sigma^2 I_n,$$

where  $\Sigma$  is a block diagonal matrix whose  $i$ th block is  $J_{n_i} J_{n_i}^\tau$  and  $J_k$  is the  $k$ -vector of ones.

Under this model,  $Z = J_n$ ,  $n = \sum_{i=1}^m n_i$ , and  $Z(Z^\tau Z)^{-1} Z^\tau = n^{-1} J_n J_n^\tau$ .

Note that

$$J_n J_n^\tau \Sigma = \begin{pmatrix} n_1 J_{n_1} J_{n_1}^\tau & n_2 J_{n_1} J_{n_2}^\tau & \cdots & n_m J_{n_1} J_{n_m}^\tau \\ n_1 J_{n_2} J_{n_1}^\tau & n_2 J_{n_2} J_{n_2}^\tau & \cdots & n_m J_{n_2} J_{n_m}^\tau \\ \cdots & \cdots & \cdots & \cdots \\ n_1 J_{n_m} J_{n_1}^\tau & n_2 J_{n_m} J_{n_2}^\tau & \cdots & n_m J_{n_m} J_{n_m}^\tau \end{pmatrix},$$

which is symmetric if and only if  $n_1 = n_2 = \cdots = n_m$ .

Since  $J_n J_n^\tau \text{Var}(\varepsilon)$  is symmetric if and only if  $J_n J_n^\tau \Sigma$  is symmetric, a necessary and sufficient condition for the LSE of  $\mu$  to be the BLUE is that all  $n_i$ 's are the same.

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In some cases, we are interested in some (not all) linear functions of  $\beta$ . For example, consider  $I^r\beta$  with  $I \in \mathcal{R}(H)$ , where  $H$  is an  $n \times p$  matrix such that  $\mathcal{R}(H) \subset \mathcal{R}(Z)$ .

### Proposition 3.4

Consider model (1) with assumption A3.

Suppose that  $H$  is a matrix such that  $\mathcal{R}(H) \subset \mathcal{R}(Z)$ .

A necessary and sufficient condition for the LSE  $I^r\hat{\beta}$  to be the BLUE of  $I^r\beta$  for any  $I \in \mathcal{R}(H)$  is  $H(Z^T Z)^- Z^T \text{Var}(\varepsilon)U = 0$ , where  $U$  is the same as that in (c) of Theorem 3.10.

### Example 3.18

Consider model (1) with assumption A3 and  $Z = (H_1 \ H_2)$ , where  $H_1^T H_2 = 0$ .

Suppose that under the reduced model

$$X = H_1\beta_1 + \varepsilon,$$

$I^r\hat{\beta}_1$  is the BLUE for any  $I^r\beta_1$ ,  $I \in \mathcal{R}(H_1)$

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### Example 3.18

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Suppose that under the reduced model

$$X = H_1 \beta_1 + \varepsilon,$$

$l^T \hat{\beta}_1$  is the BLUE for any  $l^T \beta_1$ ,  $l \in \mathcal{R}(H_1)$

In some cases, we are interested in some (not all) linear functions of  $\beta$ . For example, consider  $l^T \beta$  with  $l \in \mathcal{R}(H)$ , where  $H$  is an  $n \times p$  matrix such that  $\mathcal{R}(H) \subset \mathcal{R}(Z)$ .

### Proposition 3.4

Consider model (1) with assumption A3.

Suppose that  $H$  is a matrix such that  $\mathcal{R}(H) \subset \mathcal{R}(Z)$ .

A necessary and sufficient condition for the LSE  $l^T \hat{\beta}$  to be the BLUE of  $l^T \beta$  for any  $l \in \mathcal{R}(H)$  is  $H(Z^T Z)^- Z^T \text{Var}(\varepsilon) U = 0$ , where  $U$  is the same as that in (c) of Theorem 3.10.

### Example 3.18

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## Example 3.18 (continued)

and that under the reduced model

$$X = H_2\beta_2 + \varepsilon,$$

$I^r \widehat{\beta}_2$  is not a BLUE for some  $I^r \beta_2$ ,  $I \in \mathcal{R}(H_2)$ , where  $\beta = (\beta_1, \beta_2)$  and  $\widehat{\beta}_j$ 's are LSE's under the reduced models.

Let  $H = (H_1 \ 0)$  be  $n \times p$ .

Note that

$$H(Z^T Z)^{-1} Z^T \text{Var}(\varepsilon) U = H_1 (H_1^T H_1)^{-1} H_1^T \text{Var}(\varepsilon) U,$$

which is 0 by Theorem 3.10 for the  $U$  given in (c) of Theorem 3.10, and

$$Z(Z^T Z)^{-1} Z^T \text{Var}(\varepsilon) U = H_2 (H_2^T H_2)^{-1} H_2^T \text{Var}(\varepsilon) U,$$

which is not 0 by Theorem 3.10.

This implies that some LSE  $I^r \widehat{\beta}$  is not a BLUE of  $I^r \beta$  but  $I^r \widehat{\beta}$  is the BLUE of  $I^r \beta$  if  $I \in \mathcal{R}(H)$ .

## Diagonal $\text{Var}(\varepsilon)$

We consider model (1) with  $\text{Var}(\varepsilon)$  being a diagonal matrix whose  $i$ th diagonal element is  $\sigma_i^2$ , i.e.,  $\varepsilon_i$ 's are uncorrelated but have unequal variances.

A straightforward calculation shows that condition (e) in Theorem 3.10 holds if and only if, for all  $i \neq j$ ,  $\sigma_i^2 \neq \sigma_j^2$  only when  $h_{ij} = 0$ , where  $h_{ij}$  is the  $(i, j)$ th element of the projection matrix  $Z(Z^T Z)^- Z^T$ .

Thus, an LSE is not a BLUE in general, although it is still unbiased for estimable  $I^r \beta$ .

Suppose that the unequal variances of  $\varepsilon_i$ 's are caused by some small perturbations, i.e.,  $\varepsilon_i = e_i + u_i$ , where  $\text{Var}(e_i) = \sigma^2$ ,  $\text{Var}(u_i) = \delta_i$ , and  $e_i$  and  $u_i$  are independent so that  $\sigma_i^2 = \sigma^2 + \delta_i$  and

$$\text{Var}(I^r \hat{\beta}) = I^r (Z^T Z)^- \sum_{i=1}^n \sigma_i^2 Z_i Z_i^T (Z^T Z)^- I.$$

If  $\delta_i = 0$  for all  $i$  (no perturbations), then assumption A2 holds and  $I^r \hat{\beta}$  is the BLUE of any estimable  $I^r \beta$  with  $\text{Var}(I^r \hat{\beta}) = \sigma^2 I^r (Z^T Z)^- I$ .

## Diagonal $\text{Var}(\varepsilon)$

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If  $\delta_i = 0$  for all  $i$  (no perturbations), then assumption A2 holds and  $I^T \hat{\beta}$  is the BLUE of any estimable  $I^T \beta$  with  $\text{Var}(I^T \hat{\beta}) = \sigma^2 I^T (Z^T Z)^- I$ .

Suppose that  $0 < \delta_i \leq \sigma^2 \delta$ .

Then

$$\text{Var}(I^\tau \hat{\beta}) \leq (1 + \delta) \sigma^2 I^\tau (Z^\tau Z)^{-1} I.$$

This indicates that the LSE is robust in the sense that its variance increases slightly when there is a slight violation of the equal variance assumption (small  $\delta$ ).