Stat 709: Mathematical Statistics
Lecture 28

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Lecture 28: Asymptotic inference

Statistical inference based on asymptotic criteria and approximations is called *asymptotic statistical inference* or simply *asymptotic inference*.
We have previously considered asymptotic estimation.
We now focus on asymptotic hypothesis tests and confidence sets.

Definition 2.13

Let $X = (X_1, ..., X_n)$ be a sample from $P \in \mathscr{P}$ and $T_n(X)$ be a test for $H_0 : P \in \mathscr{P}_0$ versus $H_1 : P \in \mathscr{P}_1$.

(i) If $\limsup_n \alpha_{T_n}(P) \le \alpha$ for any $P \in \mathscr{P}_0$, then α is an *asymptotic significance level* of T_n.

(ii) If $\lim_{n\to\infty} \sup_{P\in\mathscr{P}_0} \alpha_{T_n}(P)$ exists, it is called the *limiting size* of T_n.

(iii) T_n is *consistent* iff the type II error probability converges to 0.

(iv) T_n is *Chernoff-consistent* iff T_n is consistent *and* the type I error probability converges to 0.
T_n is *strongly Chernoff-consistent* iff T_n is consistent and the limiting size of T_n is 0.

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T_n is *strongly Chernoff-consistent* iff T_n is consistent and the limiting size of T_n is 0.

Remarks

- Obviously if T_n has size (or significance level) α for all n, then its limiting size (or asymptotic significance level) is α.
- If the limiting size of T_n is $\alpha \in (0,1)$, then for any $\varepsilon > 0$, T_n has size $\alpha + \varepsilon$ for all $n \geq n_0$, where n_0 is independent of P.
  Hence T_n has level of significance $\alpha + \varepsilon$ for any $n \geq n_0$.
- However, if $\mathscr{P}_0$ is not a parametric family, it is likely that the limiting size of T_n is 1 (see, e.g., Example 2.37).
- This is the reason why we consider the weaker requirement in Definition 2.13(i).
- If T_n has asymptotic significance level α, then for any $\varepsilon > 0$, $\alpha_{T_n}(P) < \alpha + \varepsilon$ for all $n \geq n_0(P)$ but $n_0(P)$ depends on $P \in \mathscr{P}_0$; and there is no guarantee that T_n has significance level $\alpha + \varepsilon$ for any n.
- The consistency in Definition 2.13(iii) only requires that the type II error probability converge to 0.
- We may define uniform consistency to be $\lim_{n\to\infty} \sup_{P \in \mathscr{P}_1} [1 - \alpha_{T_n}(P)] = 0$, but it is not true in most problems.

Remarks

- If $\alpha \in (0,1)$ is a pre-assigned level of significance for the problem, then a consistent test T_n having asymptotic significance level α is called *asymptotically correct*, and a consistent test having limiting size α is called *strongly* asymptotically correct.
- The Chernoff-consistency (or strong Chernoff-consistency) in Definition 2.13(iv) requires that both types of error probabilities converge to 0.
- Mathematically, Chernoff-consistency is better than asymptotic correctness.
  After all, both types of error probabilities should decrease to 0 if sampling can be continued indefinitely.
- However, if α is chosen to be small enough so that error probabilities smaller than α can be practically treated as 0, then the asymptotic correctness (or strongly asymptotic correctness) is enough, and is probably preferred, since requiring an unnecessarily small type I error probability usually results in an unnecessary increase in the type II error probability.

Example 2.37

Consider the testing problem

$$H_0 : \mu \le \mu_0 \quad \text{versus} \quad H_1 : \mu > \mu_0$$

based on i.i.d. $X_1,...,X_n$ with $EX_1 = \mu \in \mathscr{R}$.

If each X_i has the $N(\mu,\sigma^2)$ distribution with a known σ^2, then the test $T_{c_\alpha} = I_{(c_\alpha,\infty)}(\bar{X})$ with $c_\alpha = \sigma z_{1-\alpha}/\sqrt{n} + \mu_0$ and $\alpha \in (0,1)$ has size α (and, therefore, limiting size α).

For any $\mu > \mu_0$,

$$1 - \alpha_{T_{c_\alpha}}(\mu) = \Phi\left(z_{1-\alpha} + \frac{\sqrt{n}(\mu_0 - \mu)}{\sigma}\right) \to 0 \qquad (1)$$

as $n \to \infty$.

This shows that T_{c_α} is consistent and, hence, is strongly asymptotically correct.

The convergence in (1) is not uniform in $\mu > \mu_0$, but is uniform in $\mu > \mu_1$ for any fixed $\mu_1 > \mu_0$.

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Example 2.37 (continued)

Since the size of T_{c_α} is α for all n, T_{c_α} is not Chernoff-consistent.
A strongly Chernoff-consistent test can be obtained as follows.
Let

$$\alpha_n = 1 - \Phi(\sqrt{n}a_n), \tag{2}$$

where a_n's are positive numbers satisfying $a_n \to 0$ and $\sqrt{n}a_n \to \infty$.
Let T_n be T_{c_α} with $\alpha = \alpha_n$ for each n.
Then, T_n has size α_n.
Since $\alpha_n \to 0$, The limiting size of T_n is 0.
On the other hand, (1) still holds with α replaced by α_n.
This follows from the fact that

$$z_{1-\alpha_n} + \frac{\sqrt{n}(\mu_0 - \mu)}{\sigma} = \sqrt{n}\left(a_n + \frac{\mu_0 - \mu}{\sigma}\right) \to -\infty$$

for any $\mu > \mu_0$.
Hence T_n is strongly Chernoff-consistent.
However, if $\alpha_n < \alpha$, then, from the left-hand side of (1),
$1 - \alpha_{T_{c_\alpha}}(\mu) < 1 - \alpha_{T_n}(\mu)$ for any $\mu > \mu_0$.

Example 2.37 (continued)

Consider now the situation where the population P is not in a parametric family.

We still assume that $\sigma^2 = \mathrm{Var}(X_i)$ is known.

Using the CLT, we can show that for $\mu > \mu_0$,

$$\lim_{n\to\infty}[1-\alpha_{T_{c_\alpha}}(\mu)] = \lim_{n\to\infty}\Phi\left(z_{1-\alpha} + \frac{\sqrt{n}(\mu_0-\mu)}{\sigma}\right) = 0,$$

i.e., T_{c_α} is still consistent.

For $\mu \le \mu_0$,

$$\lim_{n\to\infty}\alpha_{T_{c_\alpha}}(\mu) = 1 - \lim_{n\to\infty}\Phi\left(z_{1-\alpha} + \frac{\sqrt{n}(\mu_0-\mu)}{\sigma}\right),$$

which equals α if $\mu = \mu_0$ and 0 if $\mu < \mu_0$.

Thus, the asymptotic significance level of T_{c_α} is α.

Combining these two results, we know that T_{c_α} is asymptotically correct.

Example 2.37 (continued)

However, if $\mathscr{P}$ contains all possible populations on $\mathscr{R}$ with finite second moments, then one can show that the limiting size of T_{c_α} is 1 (exercise).

For α_n defined by (2), we can show that $T_n = T_{c_\alpha}$ with $\alpha = \alpha_n$ is Chernoff-consistent (exercise).

But T_n is not strongly Chernoff-consistent if $\mathscr{P}$ contains all possible populations on $\mathscr{R}$ with finite second moments.

Example

Let $(X_1, ..., X_n)$ be a random sample from the exponential distribution $E(0, \theta)$, where $\theta \in (0, \infty)$.

Consider the hypotheses $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$, where $\theta_0 > 0$ is a fixed constant.

Let $T_c = I_{(c,\infty)}(\bar{X})$, where $\bar{X}$ is the sample mean.

$\bar{X}/\theta$ has the gamma distribution with shape parameter n and scale parameter θ/n.

Example 2.37 (continued)

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Example

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Let $T_c = I_{(c,\infty)}(\bar{X})$, where $\bar{X}$ is the sample mean.
$\bar{X}/\theta$ has the gamma distribution with shape parameter n and scale parameter θ/n.

Example (continued)

Let $G_{n,\theta}$ denote the cumulative distribution function of this distribution and $c_{n,\alpha}$ be the constant satisfying $G_{n,\theta_0}(c_{n,\alpha}) = 1-\alpha$.
Then,

$$\sup_{\theta \le \theta_0} P(T_{c_{n,\alpha}} = 1) = \sup_{\theta \le \theta_0} [1 - G_{n,\theta}(c_{n,\alpha})] = 1 - G_{n,\theta_0}(c_{n,\alpha}) = \alpha,$$

i.e., the size of $T_{c_{n,\alpha}}$ is α.
Since the power of $T_{c_{n,\alpha}}$ is $P(T_{c_{n,\alpha}} = 1) = P(\bar{X} > c_{n,\alpha})$ for $\theta > \theta_0$ and, by the law of large numbers, $\bar{X} \to_p \theta$, the consistency of $T_{c_{n,\alpha}}$ follows if we can show that $\lim_n c_{n,\alpha} = \theta_0$.
By the central limit theorem, $\sqrt{n}(\bar{X} - \theta) \to_d N(0, \theta^2)$.
Hence, $\sqrt{n}(\frac{\bar{X}}{\theta} - 1) \to_d N(0,1)$.
By Pólya's theorem (Proposition 1.16),

$$\lim_n \sup_t \left| P\left(\sqrt{n}\left(\frac{\bar{X}}{\theta} - 1\right) \le t \right) - \Phi(t) \right| = 0,$$

where Φ is the cumulative distribution function of the standard normal distribution.

Example (continued)

When $\theta = \theta_0$,

$$\alpha = P(\bar{X} \geq c_{n,\alpha}) = P\left(\sqrt{n}\left(\frac{\bar{X}}{\theta_0} - 1\right) \geq \sqrt{n}\left(\frac{c_{n,\alpha}}{\theta_0} - 1\right)\right).$$

Hence

$$\lim_n \Phi\left(\sqrt{n}\left(\frac{c_{n,\alpha}}{\theta_0} - 1\right)\right) = 1 - \alpha,$$

which implies $\lim_n \sqrt{n}(\frac{c_{n,\alpha}}{\theta_0} - 1) = \Phi^{-1}(1-\alpha)$ and, thus, $\lim_n c_{n,\alpha} = \theta_0$.
Let $\{a_n\}$ be a sequence of positive numbers such that $\lim_n a_n = 0$ and $\lim_n \sqrt{n} a_n = \infty$.
Let $\alpha_n = 1 - \Phi(\sqrt{n} a_n)$ and $b_n = c_{n,\alpha_n}$.
From the previous derivation, the size of T_{b_n} is α_n, which converges to 0 as $n \to \infty$ since $\lim_n \sqrt{n} a_n = \infty$.
Using the previous argument, we can show that

$$\lim_n \left| 1 - \alpha_n - \Phi\left(\sqrt{n}\left(\frac{c_{n,\alpha_n}}{\theta_0} - 1\right)\right)\right| = 0,$$

Example 2.37 (continued)

which implies that

$$\lim_n \frac{\sqrt{n}}{\Phi^{-1}(1-\alpha_n)}\left(\frac{c_{n,\alpha_n}}{\theta_0}-1\right)=1.$$

Since $1-\alpha_n=\Phi(\sqrt{n}a_n)$, this implies that $\lim_n c_{n,\alpha_n}=\theta_0$.
Since $b_n=c_{n,\alpha_n}$, the test T_{b_n} is Chernoff-consistent.

Definition 2.14

Let $X=(X_1,...,X_n)$ be a sample from $P\in\mathscr{P}$, ϑ be a k-vector of parameters related to P, and $C(X)$ be a confidence set for ϑ.

(i) If $\liminf_n P(\vartheta\in C(X))\geq 1-\alpha$ for any $P\in\mathscr{P}$, then $1-\alpha$ is an *asymptotic significance level* of $C(X)$.

(ii) If $\lim_{n\to\infty}\inf_{P\in\mathscr{P}} P(\vartheta\in C(X))$ exists, then it is called the *limiting confidence coefficient* of $C(X)$.

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which implies that

$$\lim_n \frac{\sqrt{n}}{\Phi^{-1}(1-\alpha_n)} \left(\frac{c_{n,\alpha_n}}{\theta_0} - 1 \right) = 1.$$

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Remarks

- Note that the asymptotic significance level and limiting confidence coefficient of a confidence set are very similar to the asymptotic significance level and limiting size of a test, respectively.
- Some conclusions are also similar.
- For example, in a parametric problem one can often find a confidence set having limiting confidence coefficient $1-\alpha \in (0,1)$, which implies that for any $\varepsilon > 0$, the confidence coefficient of $C(X)$ is $1-\alpha-\varepsilon$ for all $n \geq n_0$, where n_0 is independent of P.
- In a nonparametric problem the limiting confidence coefficient of $C(X)$ might be 0, whereas $C(X)$ may have asymptotic significance level $1-\alpha \in (0,1)$, but for any fixed n, the confidence coefficient of $C(X)$ might be 0.