Stat 709: Mathematical Statistics
Lecture 21

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Motivation

A statistic $V(X)$ is *ancillary* if its distribution does not depend on the population P

$V(X)$ is *first-order ancillary* if $E[V(X)]$ is independent of P.

A trivial ancillary statistic is the constant statistic $V(X) \equiv c \in \mathscr{R}$.

If $V(X)$ is a nontrivial ancillary statistic, then $\sigma(V(X)) \subset \sigma(X)$ is a nontrivial σ-field that does not contain any information about P.

Hence, if $S(X)$ is a statistic and $V(S(X))$ is a nontrivial ancillary statistic, it indicates that $\sigma(S(X))$ contains a nontrivial σ-field that does not contain any information about P and, hence, the "data" $S(X)$ may be further reduced.

A sufficient statistic T appears to be most successful in reducing the data if no nonconstant function of T is ancillary or even first-order ancillary.

This leads to the following definition.

Finding a complete and sufficient statistic

Definition 2.6 (Completeness)

A statistic $T(X)$ is said to be *complete* for $P \in \mathscr{P}$ iff, for any Borel f, $E[f(T)] = 0$ for all $P \in \mathscr{P}$ implies $f = 0$ a.s. $\mathscr{P}$.
T is said to be *boundedly complete* iff the previous statement holds for any bounded Borel f.

Remarks

- A complete statistic is boundedly complete.
- If T is complete (or boundedly complete) and $S = \psi(T)$ for a measurable ψ, then S is complete (or boundedly complete).
- Intuitively, a complete and sufficient statistic should be minimal sufficient (Exercise 48).
- A minimal sufficient statistic is not necessarily complete; for example, the minimal sufficient statistic $(X_{(1)}, X_{(n)})$ in Example 2.13 is not complete (Exercise 47).

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Proposition 2.1

If P is in an exponential family of full rank with p.d.f.'s given by

$$f_\eta(x) = \exp\{\eta^\tau T(x) - \zeta(\eta)\}h(x),$$

then $T(X)$ is complete and sufficient for $\eta \in \Xi$.

Proof

We have shown that T is sufficient.
We now show that T is complete.
Suppose that there is a function f such that $E[f(T)] = 0$ for all $\eta \in \Xi$.
By Theorem 2.1(i),

$$\int f(t)\exp\{\eta^\tau t - \zeta(\eta)\}d\lambda = 0 \quad \text{for all } \eta \in \Xi,$$

where λ is a measure on $(\mathscr{R}^p, \mathscr{B}^p)$.

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Proof (continued)

Let η_0 be an interior point of Ξ. Then

$$\int f_+(t) e^{\eta^\tau t} d\lambda = \int f_-(t) e^{\eta^\tau t} d\lambda \quad \text{for all } \eta \in N(\eta_0), \tag{1}$$

where $N(\eta_0) = \{\eta \in \mathscr{R}^p : \|\eta - \eta_0\| < \varepsilon\}$ for some $\varepsilon > 0$.
In particular,

$$\int f_+(t) e^{\eta_0^\tau t} d\lambda = \int f_-(t) e^{\eta_0^\tau t} d\lambda = c.$$

If $c = 0$, then $f = 0$ a.e. λ.
If $c > 0$, then $c^{-1} f_+(t) e^{\eta_0^\tau t}$ and $c^{-1} f_-(t) e^{\eta_0^\tau t}$ are p.d.f.'s w.r.t. λ and result (1) implies that their m.g.f.'s are the same in a neighborhood of 0.
By Theorem 1.6(ii), $c^{-1} f_+(t) e^{\eta_0^\tau t} = c^{-1} f_-(t) e^{\eta_0^\tau t}$, i.e., $f = f_+ - f_- = 0$ a.e. λ.
Hence T is complete.

Example 2.15

Suppose that $X_1,...,X_n$ are i.i.d. random variables having the $N(\mu,\sigma^2)$ distribution, $\mu \in \mathscr{R}$, $\sigma > 0$.
From Example 2.6, the joint p.d.f. of $X_1,...,X_n$ is

$$(2\pi)^{-n/2}\exp\{\eta_1 T_1 + \eta_2 T_2 - n\zeta(\eta)\},$$

where $T_1 = \sum_{i=1}^n X_i$, $T_2 = -\sum_{i=1}^n X_i^2$, and $\eta = (\eta_1,\eta_2) = \left(\frac{\mu}{\sigma^2},\frac{1}{2\sigma^2}\right)$.
Hence, the family of distributions for $X = (X_1,...,X_n)$ is a natural exponential family of full rank ($\Xi = \mathscr{R} \times (0,\infty)$).
By Proposition 2.1, $T(X) = (T_1,T_2)$ is complete and sufficient for η.
Since there is a one-to-one correspondence between η and $\theta = (\mu,\sigma^2)$, T is also complete and sufficient for θ.
It can be shown that any one-to-one measurable function of a complete and sufficient statistic is also complete and sufficient (exercise).
Thus, $(\bar{X},S^2)$ is complete and sufficient for θ, where $\bar{X}$ and S^2 are the sample mean and sample variance, respectively.

Example 2.16

Let $X_1,...,X_n$ be i.i.d. random variables from P_θ, the uniform distribution $U(0,\theta)$, $\theta > 0$.

The largest order statistic, $X_{(n)}$, is complete and sufficient for $\theta \in (0,\infty)$.

The sufficiency of $X_{(n)}$ follows from the fact that the joint Lebesgue p.d.f. of $X_1,...,X_n$ is $\theta^{-n} I_{(0,\theta)}(x_{(n)})$.

From Example 2.9, $X_{(n)}$ has the Lebesgue p.d.f. $(nx^{n-1}/\theta^n) I_{(0,\theta)}(x)$.

Let f be a Borel function on $[0,\infty)$ such that $E[f(X_{(n)})] = 0$ for all $\theta > 0$.

Then

$$\int_0^\theta f(x) x^{n-1} dx = 0 \quad \text{for all } \theta > 0.$$

Let $G(\theta)$ be the left-hand side of the previous equation.

Applying the result of differentiation of an integral (see, e.g., Royden (1968, §5.3)), we obtain that $G'(\theta) = f(\theta)\theta^{n-1}$ a.e. m_+, where m_+ is the Lebesgue measure on $([0,\infty), \mathscr{B}_{[0,\infty)})$.

Since $G(\theta) = 0$ for all $\theta > 0$, $f(\theta)\theta^{n-1} = 0$ a.e. m_+ and, hence, $f(x) = 0$ a.e. m_+.

Therefore, $X_{(n)}$ is complete and sufficient for $\theta \in (0,\infty)$.

Example 2.17

In Example 2.12, we showed that the order statistics $T(X) = (X_{(1)}, ..., X_{(n)})$ of i.i.d. random variables $X_1, ..., X_n$ is sufficient for $P \in \mathscr{P}$, where $\mathscr{P}$ is the family of distributions on $\mathscr{R}$ having Lebesgue p.d.f.'s.

We now show that $T(X)$ is also complete for $P \in \mathscr{P}$.

Let $\mathscr{P}_0$ be the family of Lebesgue p.d.f.'s of the form

$$f(x) = C(\theta_1, ..., \theta_n) \exp\{-x^{2n} + \theta_1 x + \theta_2 x^2 + \cdots + \theta_n x^n\},$$

where $\theta_j \in \mathscr{R}$ and $C(\theta_1, ..., \theta_n)$ is a normalizing constant such that $\int f(x)dx = 1$.

Then $\mathscr{P}_0 \subset \mathscr{P}$ and $\mathscr{P}_0$ is an exponential family of full rank.

Note that the joint distribution of $X = (X_1, ..., X_n)$ is also in an exponential family of full rank.

Thus, by Proposition 2.1, $U = (U_1, ..., U_n)$ is a complete statistic for $P \in \mathscr{P}_0$, where $U_j = \sum_{i=1}^n X_i^j$.

Since a.s. $\mathscr{P}_0$ implies a.s. $\mathscr{P}$, $U(X)$ is also complete for $P \in \mathscr{P}$.

Example 2.17 (continued)

The result follows if we can show that there is a one-to-one correspondence between $T(X)$ and $U(X)$.
Let $V_1 = \sum_{i=1}^n X_i$, $V_2 = \sum_{i<j} X_i X_j$, $V_3 = \sum_{i<j<k} X_i X_j X_k$,..., $V_n = X_1 \cdots X_n$.
From the identities

$$U_k - V_1 U_{k-1} + V_2 U_{k-2} - \cdots + (-1)^{k-1} V_{k-1} U_1 + (-1)^k k V_k = 0,$$

$k = 1,...,n$, there is a one-to-one correspondence between $U(X)$ and $V(X) = (V_1,...,V_n)$.
From the identity

$$(t - X_1) \cdots (t - X_n) = t^n - V_1 t^{n-1} + V_2 t^{n-2} - \cdots + (-1)^n V_n,$$

there is a one-to-one correspondence between $V(X)$ and $T(X)$.
This completes the proof and, hence, $T(X)$ is sufficient and complete for $P \in \mathscr{P}$.
In fact, both $U(X)$ and $V(X)$ are sufficient and complete for $P \in \mathscr{P}$.

The relationship between an ancillary statistic and a complete and sufficient statistic is characterized in the following result.

Theorem 2.4 (Basu's theorem)

Let V and T be two statistics of X from a population $P \in \mathscr{P}$.
If V is ancillary and T is boundedly complete and sufficient for $P \in \mathscr{P}$, then V and T are independent w.r.t. any $P \in \mathscr{P}$.

Proof

Let B be an event on the range of V.
Since V is ancillary, $P(V^{-1}(B))$ is a constant.
As T is sufficient, $E[I_B(V)|T]$ is a function of T (not dependent on P).
Because

$$E\{E[I_B(V)|T] - P(V^{-1}(B))\} = 0 \quad \text{for all } P \in \mathscr{P},$$

by the bounded completeness of T,

$$P(V^{-1}(B)|T) = E[I_B(V)|T] = P(V^{-1}(B)) \quad \text{a.s. } \mathscr{P}$$

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Proof (continued)

Let A be an event on the range of T.
Then

$$P(T^{-1}(A)\cap V^{-1}(B)) = E\{E[I_A(T)I_B(V)|T]\} = E\{I_A(T)E[I_B(V)|T]\}$$
$$= E\{I_A(T)P(V^{-1}(B))\} = P(T^{-1}(A))P(V^{-1}(B)).$$

Hence T and V are independent w.r.t. any $P \in \mathscr{P}$.

Remark

Basu's theorem is useful in proving the independence of two statistics.

Example 2.18

Suppose that $X_1,...,X_n$ are i.i.d. random variables having the $N(\mu,\sigma^2)$ distribution, with $\mu \in \mathscr{R}$ and a known $\sigma > 0$.
It can be easily shown that the family $\{N(\mu,\sigma^2) : \mu \in \mathscr{R}\}$ is an exponential family of full rank with natural parameter $\eta = \mu/\sigma^2$.
By Proposition 2.1, the sample mean $\bar{X}$ is complete and sufficient for η (and μ).

Proof (continued)

Let A be an event on the range of T.
Then

$$P(T^{-1}(A) \cap V^{-1}(B)) = E\{E[I_A(T)I_B(V)|T]\} = E\{I_A(T)E[I_B(V)|T]\}$$
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Proof (continued)

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Example 2.18 (continued)

Let S^2 be the sample variance.
Since $S^2 = (n-1)^{-1}\sum_{i=1}^n (Z_i - \bar{Z})^2$, where $Z_i = X_i - \mu$ is $N(0,\sigma^2)$ and $\bar{Z} = n^{-1}\sum_{i=1}^n Z_i$, S^2 is an ancillary statistic (σ^2 is known).
By Basu's theorem, $\bar{X}$ and S^2 are independent w.r.t. $N(\mu,\sigma^2)$ with $\mu \in \mathscr{R}$.
Since σ^2 is arbitrary, $\bar{X}$ and S^2 are independent w.r.t. $N(\mu,\sigma^2)$ for any $\mu \in \mathscr{R}$ and $\sigma^2 > 0$.

Using the independence of $\bar{X}$ and S^2, we now show that $(n-1)S^2/\sigma^2$ has the chi-square distribution χ^2_{n-1}.
Note that

$$n\left(\frac{\bar{X}-\mu}{\sigma}\right)^2 + \frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^n \left(\frac{X_i-\mu}{\sigma}\right)^2.$$

From the properties of the normal distributions, $n(\bar{X}-\mu)^2/\sigma^2$ has the chi-square distribution χ^2_1 with the m.g.f. $(1-2t)^{-1/2}$ and $\sum_{i=1}^n (X_i-\mu)^2/\sigma^2$ has the chi-square distribution χ^2_n with the m.g.f. $(1-2t)^{-n/2}$, $t < 1/2$.

Example 2.18 (continued)

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Since σ^2 is arbitrary, $\bar{X}$ and S^2 are independent w.r.t. $N(\mu,\sigma^2)$ for any $\mu \in \mathscr{R}$ and $\sigma^2 > 0$.

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Example 2.18 (continued)

By the independence of $\bar{X}$ and S^2, the m.g.f. of $(n-1)S^2/\sigma^2$ is

$$(1-2t)^{-n/2}/(1-2t)^{-1/2} = (1-2t)^{-(n-1)/2}$$

for $t < 1/2$.
This is the m.g.f. of the chi-square distribution χ^2_{n-1} and, therefore, the result follows.