Stat 709: Mathematical Statistics
Lecture 9

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Definition 1.7.

Let $(\Omega, \mathscr{F}, P)$ be a probability space.

(i) Let $\mathscr{C}$ be a collection of subsets in $\mathscr{F}$.
Events in $\mathscr{C}$ are said to be *independent* iff for any positive integer n and distinct events $A_1, ..., A_n$ in $\mathscr{C}$,

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2)\cdots P(A_n).$$

(ii) Collections $\mathscr{C}_i \subset \mathscr{F}$, $i \in \mathscr{I}$ (an index set that can be uncountable), are said to be independent iff events in any collection of the form $\{A_i \in \mathscr{C}_i : i \in \mathscr{I}\}$ are independent.

(iii) Random elements X_i, $i \in \mathscr{I}$, are said to be independent iff $\sigma(X_i)$, $i \in \mathscr{I}$, are independent.

Lemma 1.3 (a useful result for checking the independence of σ-fields)

Let $\mathscr{C}_i$, $i \in \mathscr{I}$, be independent collections of events.
If each $\mathscr{C}_i$ is a π-system ($A \in \mathscr{C}_i$ and $B \in \mathscr{C}_i$ implies $A \cap B \in \mathscr{C}_i$), then $\sigma(\mathscr{C}_i)$, $i \in \mathscr{I}$, are independent.

Facts

- Random variables X_i, $i = 1,...,k$, are independent according to Definition 1.7 iff

$$F_{(X_1,...,X_k)}(x_1,...,x_k) = F_{X_1}(x_1)\cdots F_{X_k}(x_k), \quad (x_1,...,x_k) \in \mathscr{R}^k$$

  Take $\mathscr{C}_i = \{(a,b] : a \in \mathscr{R}, b \in \mathscr{R}\}$, $i = 1,...,k$

- If X and Y are independent random vectors, then so are $g(X)$ and $h(Y)$ for Borel functions g and h.
- Two events A and B are independent iff $P(B|A) = P(B)$, which means that A provides no information about the probability of the occurrence of B.

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- Random variables X_i, $i = 1, ..., k$, are independent according to Definition 1.7 iff

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Proposition 1.11

Let X be a random variable with $E|X| < \infty$ and let Y_i be random k_i-vectors, $i = 1,2$.
Suppose that (X,Y_1) and Y_2 are independent.
Then

$$E[X|(Y_1,Y_2)] = E(X|Y_1) \quad \text{a.s.}$$

Proof

First, $E(X|Y_1)$ is Borel on $(\Omega, \sigma(Y_1,Y_2))$, since $\sigma(Y_1) \subset \sigma(Y_1,Y_2)$.
Next, we need to show that for any Borel set $B \in \mathscr{B}^{k_1+k_2}$,

$$\int_{(Y_1,Y_2)^{-1}(B)} X dP = \int_{(Y_1,Y_2)^{-1}(B)} E(X|Y_1) dP.$$

If $B = B_1 \times B_2$, where $B_i \in \mathscr{B}^{k_i}$, then

$$(Y_1,Y_2)^{-1}(B) = Y_1^{-1}(B_1) \cap Y_2^{-1}(B_2)$$

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Proof (continued)

and

$$\int_{Y_1^{-1}(B_1)\cap Y_2^{-1}(B_2)} E(X|Y_1)dP = \int I_{Y_1^{-1}(B_1)} I_{Y_2^{-1}(B_2)} E(X|Y_1)dP$$
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where the second and the next to last equalities follow the independence of (X, Y_1) and Y_2, and the third equality follows from the fact that $E(X|Y_1)$ is the conditional expectation of X given Y_1.
This shows that the result for $B = B_1 \times B_2$.
Note that $\mathscr{B}^{k_1} \times \mathscr{B}^{k_2}$ is a π-system.

Proof (continued)

We can show that the following collection is a λ-system:

$$\mathscr{H} = \left\{ B \subset \mathscr{R}^{k_1+k_2} : \int_{(Y_1,Y_2)^{-1}(B)} X dP = \int_{(Y_1,Y_2)^{-1}(B)} E(X|Y_1) dP \right\}$$

Since we have already shown that $\mathscr{B}^{k_1} \times \mathscr{B}^{k_2} \subset \mathscr{H}$, $\mathscr{B}^{k_1+k_2} = \sigma(\mathscr{B}^{k_1} \times \mathscr{B}^{k_2}) \subset \mathscr{H}$ and thus the result follows.

Remarks

- The result in Proposition 1.11 still holds if X is replaced by $h(X)$ for any Borel h and, hence,

$$P(A|Y_1, Y_2) = P(A|Y_1) \text{ a.s. for any } A \in \sigma(X), \tag{1}$$

  if (X, Y_1) and Y_2 are independent.
- We say that given Y_1, X and Y_2 are *conditionally independent* iff (1) holds.
- Proposition 1.11 can be stated as: if Y_2 and (X, Y_1) are independent, then given Y_1, X and Y_2 are conditionally independent.

Proof (continued)

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Conditional distribution

For random vectors X and Y, is $P[X^{-1}(B)|Y=y]$ a probability measure for given y?
Problem: $P[X^{-1}(B)|Y=y]$ is defined a.s.

Theorem 1.7(i) (Existence of conditional distributions)

Let X be a random n-vector on a probability space $(\Omega,\mathscr{F},P)$ and $\mathscr{A}$ be a sub-σ-field of $\mathscr{F}$.
Then there exists a function $P(B,\omega)$ on $\mathscr{B}^n\times\Omega$ such that

(a) $P(B,\omega)=P[X^{-1}(B)|\mathscr{A}]$ a.s. for any fixed $B\in\mathscr{B}^n$, and

(b) $P(\cdot,\omega)$ is a probability measure on $(\mathscr{R}^n,\mathscr{B}^n)$ for any fixed $\omega\in\Omega$.

Let Y be measurable from $(\Omega,\mathscr{F},P)$ to $(\Lambda,\mathscr{G})$.
Then there exists $P_{X|Y}(B|y)$ such that

(a) $P_{X|Y}(B|y)=P[X^{-1}(B)|Y=y]$ a.s. P_Y for any fixed $B\in\mathscr{B}^n$, and

(b) $P_{X|Y}(\cdot|y)$ is a probability measure on $(\mathscr{R}^n,\mathscr{B}^n)$ for any fixed $y\in\Lambda$.

Furthermore, if $E|g(X,Y)|<\infty$ with a Borel function g, then

$$E[g(X,Y)|Y=y]=E[g(X,y)|Y=y]=\int_{\mathscr{R}^n} g(x,y)dP_{X|Y}(x|y) \quad \text{a.s. } P_Y.$$

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Theorem 1.7(ii)

Let $(\Lambda, \mathscr{G}, P_1)$ be a probability space.
Suppose that P_2 is a function from $\mathscr{B}^n \times \Lambda$ to $\mathscr{R}$ and satisfies

(a) $P_2(\cdot, y)$ is a probability measure on $(\mathscr{R}^n, \mathscr{B}^n)$ for any $y \in \Lambda$, and
(b) $P_2(B, \cdot)$ is Borel for any $B \in \mathscr{B}^n$.

Then there is a unique probability measure P on $(\mathscr{R}^n \times \Lambda, \sigma(\mathscr{B}^n \times \mathscr{G}))$ such that, for $B \in \mathscr{B}^n$ and $C \in \mathscr{G}$,

$$P(B \times C) = \int_C P_2(B, y) dP_1(y). \qquad (2)$$

Furthermore, if $(\Lambda, \mathscr{G}) = (\mathscr{R}^m, \mathscr{B}^m)$, and $X(x,y) = x$ and $Y(x,y) = y$ define the coordinate random vectors, then $P_Y = P_1$, $P_{X|Y}(\cdot|y) = P_2(\cdot, y)$, and the probability measure in (2) is the joint distribution of (X, Y), which has the following joint c.d.f.:

$$F(x,y) = \int_{(-\infty, y]} P_{X|Y}((-\infty, x]|z) dP_Y(z), \quad x \in \mathscr{R}^n, y \in \mathscr{R}^m, \qquad (3)$$

where $(-\infty, a]$ denotes $(-\infty, a_1] \times \cdots \times (-\infty, a_k]$ for $a = (a_1, ..., a_k)$.

Conditional distribution

For a fixed y, $P_{X|Y=y} = P_{X|Y}(\cdot|y)$ is called the conditional distribution of X given $Y = y$.

Two-stage experiment theorem

If $Y \in \mathscr{R}^m$ is selected in stage 1 of an experiment according to its marginal distribution $P_Y = P_1$, and X is chosen afterward according to a distribution $P_2(\cdot, y)$, then the combined two-stage experiment produces a jointly distributed pair (X, Y) with distribution $P_{(X,Y)}$ given by (2) and $P_{X|Y=y} = P_2(\cdot, y)$.

This provides a way of generating dependent random variables.

Example 1.23

A market survey is conducted to study whether a new product is preferred over the product currently available in the market (old product).

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Example 1.23 (continued)

The survey is conducted by mail.
Questionnaires are sent along with the sample products (both new and old) to N customers randomly selected from a population, where N is a positive integer.
Each customer is asked to fill out the questionnaire and return it.
Responses from customers are either 1 (new is better than old) or 0 (otherwise).
Some customers, however, do not return the questionnaires.
Let X be the number of ones in the returned questionnaires.
What is the distribution of X?

If every customer returns the questionnaire, then (from elementary probability) X has the binomial distribution $Bi(p, N)$ in Table 1.1 (assuming that the population is large enough so that customers respond independently), where $p \in (0, 1)$ is the overall rate of customers who prefer the new product.

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Example 1.23 (continued)

Now, let Y be the number of customers who respond.
Then Y is random.
Suppose that customers respond independently with the same probability $\pi \in (0,1)$.
Then P_Y is the binomial distribution $Bi(\pi, N)$.
Given $Y = y$ (an integer between 0 and N), $P_{X|Y=y}$ is the binomial distribution $Bi(p, y)$ if $y \geq 1$ and the point mass at 0 if $y = 0$.
Using (3) and the fact that binomial distributions have p.d.f.'s w.r.t. counting measure, we obtain that the joint c.d.f. of (X, Y) is

$$\begin{aligned} F(x,y) &= \sum_{k=0}^{y} P_{X|Y=k}((-\infty, x]) \binom{N}{k} \pi^k (1-\pi)^{N-k} \\ &= \sum_{k=0}^{y} \sum_{j=0}^{\min\{x,k\}} \binom{k}{j} p^j (1-p)^{k-j} \binom{N}{k} \pi^k (1-\pi)^{N-k} \end{aligned}$$

for $x = 0, 1, ..., y$, $y = 0, 1, ..., N$.
The marginal c.d.f. $F_X(x) = F(x, \infty) = F(x, N)$.

Example 1.23 (continued)

The p.d.f. of X w.r.t. counting measure is

$$f_X(x) = \sum_{k=x}^{N} \binom{k}{x} p^x (1-p)^{k-x} \binom{N}{k} \pi^k (1-\pi)^{N-k}$$

$$= \binom{N}{x} (\pi p)^x (1-\pi p)^{N-x} \sum_{k=x}^{N} \binom{N-x}{k-x} \left(\frac{\pi - \pi p}{1-\pi p}\right)^{k-x} \left(\frac{1-\pi}{1-\pi p}\right)^{N-k}$$

$$= \binom{N}{x} (\pi p)^x (1-\pi p)^{N-x}$$

for $x = 0, 1, \ldots, N$.

It turns out that the marginal distribution of X is the binomial distribution $Bi(\pi p, N)$.