Stat 709: Mathematical Statistics
Lecture 4

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Lecture 4: Convergence theorems, change of variable, and Fubini's theorem

Exchange limit and integration

$\{f_n : n = 1, 2, ...\}$: a sequence of Borel functions.
Can we exchange the limit and integration, i.e.,

$$\int \lim_{n\to\infty} f_n d\nu = \lim_{n\to\infty} \int f_n d\nu ?$$

Example 1.7

Consider $(\mathscr{R}, \mathscr{B})$ and the Lebesgue measure.
Define $f_n(x) = nI_{[0,n^{-1}]}(x)$, $n = 1, 2, ...$
Then $\lim_{n\to\infty} f_n(x) = 0$ for all x but $x = 0$.
Since the Lebesgue measure of a single point set is 0, $\lim_{n\to\infty} f_n(x) = 0$ a.e. and $\int \lim_{n\to\infty} f_n(x)dx = 0$.
On the other hand, $\int f_n(x)dx = 1$ for any n and, hence, $\lim_{n\to\infty} \int f_n(x)dx = 1$.

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Sufficient conditions

Theorem 1.1

Let $f_1, f_2, \ldots$ be a sequence of Borel functions on $(\Omega, \mathscr{F}, \nu)$.

(i) (Fatou's lemma). If $f_n \geq 0$, then $\int \liminf_n f_n d\nu \leq \liminf_n \int f_n d\nu$.

(ii) (Dominated convergence theorem). If $\lim_{n\to\infty} f_n = f$ a.e. and there exists an integrable function g such that $|f_n| \leq g$ a.e., then $\int \lim_{n\to\infty} f_n d\nu = \lim_{n\to\infty} \int f_n d\nu$.

(iii) (Monotone convergence theorem). If $0 \leq f_1 \leq f_2 \leq \cdots$ and $\lim_{n\to\infty} f_n = f$ a.e., then $\int \lim_{n\to\infty} f_n d\nu = \lim_{n\to\infty} \int f_n d\nu$.

Note

(a) To apply each part of the theorem, you need to check the conditions.

(b) If the conditions are not satisfied, you cannot apply the theorem, but it does not imply that you cannot exchange the limit and integration.

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Partial proof of Theorem 1.1

Part (i) and part (iii) are equivalent (exercise)

See the textbook for a proof of part (iii).

We now prove part (ii) (the DCT) using Faton's lemma (part (iii))

By the condition, $g + f_n \geq 0$ and $g - f_n \geq 0$
By Faton's lemma and the fact that $\lim_n f_n = f$,

$$\int (g+f)d\nu = \int \liminf_n (g+f_n)d\nu \leq \liminf_n \int (g+f_n)d\nu$$

$$\int (g-f)d\nu = \int \liminf_n (g-f_n)d\nu \leq \liminf_n \int (g-f_n)d\nu$$

The last expression is the same as

$$\int (f-g)d\nu \geq \limsup_n \int (f_n-g)d\nu$$

Since g is integrable, all integrals are finite and we can cancel $\int gd\nu$ in the above inequalities.

Then

$$\int fd\nu \leq \liminf_n \int f_n d\nu \leq \limsup_n \int f_n d\nu \leq \int fd\nu$$

Example

Let $f_n(x) = \frac{n}{x+n}$, $x \in \Omega = [0,1]$, $n = 1,2,...$
Then $\lim_n f_n(x) = 1$.
To apply the DCT, note that $0 \le f_n(x) \le 1$.
To apply the MCT, note that $0 \le f_n(x) \le f_{n+1}(x)$.
Hence, $\lim_n \int f_n(x)dx = \int \lim_n f_n(x)dx = \int dx = 1$.

Example 1.8 (Interchange of differentiation and integration)

Let $(\Omega, \mathscr{F}, \nu)$ be a measure space and, for any fixed $\theta \in \mathscr{R}$, let $f(\omega,\theta)$ be a Borel function on Ω.
Suppose that $\partial f(\omega,\theta)/\partial\theta$ exists a.e. for $\theta \in (a,b) \subset \mathscr{R}$ and that $|\partial f(\omega,\theta)/\partial\theta| \le g(\omega)$ a.e., where g is an integrable function on Ω.
Then, for each $\theta \in (a,b)$, $\partial f(\omega,\theta)/\partial\theta$ is integrable and, by Theorem 1.1(ii),

$$\frac{d}{d\theta}\int f(\omega,\theta)d\nu = \int \frac{\partial f(\omega,\theta)}{\partial\theta}d\nu.$$

Example

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Theorem 1.2 (Change of variables)

Let f be measurable from $(\Omega, \mathscr{F}, \nu)$ to $(\Lambda, \mathscr{G})$ and g be Borel on $(\Lambda, \mathscr{G})$. Then

$$\int_\Omega g \circ f d\nu = \int_\Lambda g d(\nu \circ f^{-1}),$$

i.e., if either integral exists, then so does the other, and the two are the same.

Remarks

- For Riemann integrals, $\int g(y)dy = \int g(f(x))f'(x)dx$, $y = f(x)$.
- For a random variable X on $(\Omega, \mathscr{F}, P)$, $EX = \int_\Omega X dP = \int_{\mathscr{R}} x dP_X$, $P_X = P \circ X^{-1}$
- Let Y be a random vector from Ω to $\mathscr{R}^k$ and g be Borel on $\mathscr{R}^k$.
  - Example: $Y = (X_1, X_2)$ and $g(Y) = X_1 + X_2$.
  - $E(X_1 + X_2) = EX_1 + EX_2$ (why?) $= \int_{\mathscr{R}} x dP_{X_1} + \int_{\mathscr{R}} x dP_{X_2}$.
  - We need to handle two integrals involving P_{X_1} and P_{X_2}.
  - On the other hand, $E(X_1 + X_2) = \int_{\mathscr{R}} x dP_{X_1+X_2}$ involving one integral w.r.t. $P_{X_1+X_2}$, which is not easy to obtain unless we have some knowledge about the joint c.d.f. of (X_1, X_2).

Iterated integration on a product space

Theorem 1.3 (Fubini's theorem)

Let ν_i be a σ-finite measure on $(\Omega_i, \mathscr{F}_i)$, $i = 1, 2$, and let f be a Borel function on $\prod_{i=1}^{2}(\Omega_i, \mathscr{F}_i)$.
Suppose that either $f \geq 0$ or $\int |f| \nu_1 \times \nu_2 < \infty$.
Then

$$g(\omega_2) = \int_{\Omega_1} f(\omega_1, \omega_2) d\nu_1$$

exists a.e. ν_2 and defines a Borel function on Ω_2 whose integral w.r.t. ν_2 exists, and

$$\int_{\Omega_1 \times \Omega_2} f(\omega_1, \omega_2) d\nu_1 \times \nu_2 = \int_{\Omega_2} \left[\int_{\Omega_1} f(\omega_1, \omega_2) d\nu_1 \right] d\nu_2.$$

Note

Extensions to $\prod_{i=1}^{k}(\Omega_i, \mathscr{F}_i)$ is straightforward.

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Fubini's theorem is *very useful* in

1. evaluating multi-dimensional integrals (exchanging the order of integrals);
2. proving a function is measurable;
3. proving some results by relating a one dimensional integral to a multi-dimensional integral

Example 1.9

Let $\Omega_1 = \Omega_2 = \{0, 1, 2, ...\}$, and $\nu_1 = \nu_2$ be the counting measure.
A function f on $\Omega_1 \times \Omega_2$ defines a double sequence.
If either $f \geq 0$ or $\int |f| d\nu_1 \times \nu_2 < \infty$, then

$$\int f d\nu_1 \times \nu_2 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} f(i,j) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} f(i,j)$$

(by Theorem 1.3 and Example 1.5).
Thus, a double series can be summed in either order, if it is summable or $f \geq 0$.

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Thus, a double series can be summed in either order, if it is summable or $f \geq 0$.

Example: Exercise 47

Let X and Y be random variables such that the joint c.d.f. of (X,Y) is $F_X(x)F_Y(y)$, where F_X and F_Y are marginal c.d.f.'s.
Let $Z = X + Y$.
We want to show that

$$F_Z(z) = \int F_Y(z-x)dF_X(x).$$

Note that

$$\begin{aligned} F_Z(z) &= \int_{x+y\le z} dF_X(x)dF_Y(y) \\ &= \int \left(\int_{y\le z-x} dF_Y(y) \right) dF_X(x) \\ &= \int F_Y(z-x)dF_X(x), \end{aligned}$$

where the second equality follows from Fubini's theorem.