

Stat 709: Mathematical Statistics

Lecture 3

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Lecture 3: Integration

Integration is a type of "average".

Definition 1.4

- (a) The integral of a nonnegative simple function φ w.r.t. ν is defined as

$$\int \varphi d\nu = \sum_{i=1}^k a_i \nu(A_i).$$

- (b) Let f be a nonnegative Borel function and let \mathcal{S}_f be the collection of all nonnegative simple functions satisfying $\varphi(\omega) \leq f(\omega)$ for any $\omega \in \Omega$. The integral of f w.r.t. ν is defined as

$$\int f d\nu = \sup \left\{ \int \varphi d\nu : \varphi \in \mathcal{S}_f \right\}.$$

(Hence, for any Borel function $f \geq 0$, there exists a sequence of simple functions $\varphi_1, \varphi_2, \dots$ such that $0 \leq \varphi_i \leq f$ for all i and $\lim_{n \rightarrow \infty} \int \varphi_n d\nu = \int f d\nu$.)

Definition 1.4 (continued)

- (c) Let f be a Borel function,

$$f_+(\omega) = \max\{f(\omega), 0\}$$

be the positive part of f , and

$$f_-(\omega) = \max\{-f(\omega), 0\}$$

be the negative part of f . (Note that f_+ and f_- are nonnegative Borel functions, $f(\omega) = f_+(\omega) - f_-(\omega)$, and $|f(\omega)| = f_+(\omega) + f_-(\omega)$.) We say that $\int f d\nu$ exists if and only if at least one of $\int f_+ d\nu$ and $\int f_- d\nu$ is finite, in which case

$$\int f d\nu = \int f_+ d\nu - \int f_- d\nu.$$

- (d) When both $\int f_+ d\nu$ and $\int f_- d\nu$ are finite, we say that f is integrable. Let A be a measurable set and I_A be its indicator function. The integral of f over A is defined as

$$\int_A f d\nu = \int I_A f d\nu.$$

Remark

A Borel function f is integrable if and only if $|f|$ is integrable.

Notation for integrals

- $\int f d\nu = \int_{\Omega} f d\nu = \int f(\omega) d\nu = \int f(\omega) d\nu(\omega) = \int f(\omega) \nu(d\omega)$.
- In probability and statistics, $\int X dP = EX = E(X)$ and is called the *expectation* or *expected value* of X .
- If F is the c.d.f. of P on $(\mathcal{R}^k, \mathcal{B}^k)$, $\int f(x) dP = \int f(x) dF(x) = \int f dF$.

Extended set

For convenience, we define the integral of a measurable function f from $(\Omega, \mathcal{F}, \nu)$ to $(\bar{\mathcal{R}}, \bar{\mathcal{B}})$, where $\bar{\mathcal{R}} = \mathcal{R} \cup \{-\infty, \infty\}$,

$\bar{\mathcal{B}} = \sigma(\mathcal{B} \cup \{\{\infty\}, \{-\infty\}\})$.

Let $A_+ = \{f = \infty\}$ and $A_- = \{f = -\infty\}$.

If $\nu(A_+) = 0$, we define $\int f_+ d\nu$ to be $\int I_{A_+^c} f_+ d\nu$; otherwise $\int f_+ d\nu = \infty$.

$\int f_- d\nu$ is similarly defined.

If at least one of $\int f_+ d\nu$ and $\int f_- d\nu$ is finite, then

$\int f d\nu = \int f_+ d\nu - \int f_- d\nu$ is well defined.

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Example 1.5

Let Ω be a countable set, \mathcal{F} be all subsets of Ω , and ν be the counting measure.

For any Borel function f ,

$$\int f d\nu = \sum_{\omega \in \Omega} f(\omega).$$

Example 1.6.

If $\Omega = \mathcal{R}$ and ν is the Lebesgue measure, then the Lebesgue integral of f over an interval $[a, b]$ is written as

$$\int_{[a,b]} f(x) dx = \int_a^b f(x) dx,$$

which agrees with the Riemann integral in calculus when the latter is well defined.

However, there are functions for which the Lebesgue integrals are defined but not the Riemann integrals.

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Proposition 1.5

Let $(\Omega, \mathcal{F}, \nu)$ be a measure space and f and g be Borel functions.

- (i) If $\int f d\nu$ exists and $a \in \mathcal{R}$, then $\int (af) d\nu$ exists and is equal to $a \int f d\nu$.
- (ii) If both $\int f d\nu$ and $\int g d\nu$ exist and $\int f d\nu + \int g d\nu$ is well defined, then $\int (f+g) d\nu$ exists and is equal to $\int f d\nu + \int g d\nu$.

Remarks.

- Integration is a linear operator: $\int (af + bg) d\nu = a \int f d\nu + b \int g d\nu$
- For integrals in calculus, properties such as $\int (af) d\nu = a \int f d\nu$ and $\int (f+g) d\nu = \int f d\nu + \int g d\nu$ are obvious. However, the proof of them are complicated for integrals defined on general measure spaces.
- It is often a good idea to break the proof into several steps: simple functions, nonnegative functions, and then general functions.

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Proof of Proposition 1.5

We prove (i) (the proof of (ii) is left as an exercise).

If $a = 0$, then $\int (af) dv = \int 0 dv = 0 = a \int f dv$.

Suppose that $a > 0$.

If f is simple and ≥ 0 , then af is also simple and ≥ 0 and $a \int f dv = \int (af) dv$ follows from the definition of integration.

For $a > 0$ and a general $f \geq 0$,

$$\begin{aligned} \int (af) dv &= \sup \left\{ \int \phi dv : \phi \in \mathcal{S}_{af} \right\} \\ &= \sup \left\{ \int a\phi dv : \phi \in \mathcal{S}_f \right\} \quad (\phi = \phi/a) \\ &= \sup \left\{ a \int \phi dv : \phi \in \mathcal{S}_f \right\} \\ &= a \sup \left\{ \int \phi dv : \phi \in \mathcal{S}_f \right\} \\ &= a \int f dv \end{aligned}$$

Proof (continued)

For $a > 0$ and general f , since $\int f dv$ exists,

$$\begin{aligned} a \int f dv &= a \left(\int f_+ dv - \int f_- dv \right) \\ &= a \int f_+ dv - a \int f_- dv \\ &= \int af_+ dv - \int af_- dv \\ &= \int (af)_+ dv - \int (af)_- dv \\ &= \int (af) dv \end{aligned}$$

Proof (continued)

For $a < 0$, $af = |a|(-f)$ and

$$\begin{aligned} a \int f dv &= |a| \left(- \int f dv \right) \\ &= |a| \left[- \left(\int f_+ dv - \int f_- dv \right) \right] \\ &= |a| \left(\int f_- dv - \int f_+ dv \right) \\ &= |a| \left(\int (-f)_+ dv - \int (-f)_- dv \right) \\ &= |a| \int (-f) dv \\ &= \int |a|(-f) dv \\ &= \int (af) dv \end{aligned}$$

a.e. and a.s. statements

A statement holds a.e. ν (or simply a.e.) if it holds for all ω in N^c with $\nu(N) = 0$.

If ν is a probability, then a.e. may be replaced by a.s.

Proposition 1.6

Let $(\Omega, \mathcal{F}, \nu)$ be a measure space and f and g be Borel functions.

- (i) If $f \leq g$ a.e., then $\int f d\nu \leq \int g d\nu$, provided that the integrals exist.
- (ii) If $f \geq 0$ a.e. and $\int f d\nu = 0$, then $f = 0$ a.e.

Proof

(i) Since $f - g \geq 0$, by the definition of integration,

$$\int (f - g) d\nu \geq 0.$$

By Proposition 1.5(ii),

$$\int f d\nu - \int g d\nu = \int (f - g) d\nu \geq 0.$$

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Proof (continued)

(ii) Let $A = \{f > 0\}$ and $A_n = \{f \geq n^{-1}\}$, $n = 1, 2, \dots$

Then $A_n \subset A$ for any n and $\lim_{n \rightarrow \infty} A_n = \cup A_n = A$ (why?).

By Proposition 1.1(iii), $\lim_{n \rightarrow \infty} v(A_n) = v(A)$.

Using part (i) and Proposition 1.5, we obtain that

$$n^{-1}v(A_n) = \int n^{-1}I_{A_n} dv \leq \int fI_{A_n} dv \leq \int f dv = 0$$

for any n . Hence $v(A) = 0$ and $f = 0$ a.e.

Consequences

- $|\int f dv| \leq \int |f| dv$
- If $f \geq 0$ a.e., then $\int f dv \geq 0$
- If $f = g$ a.e., then $\int f dv = \int g dv$.

Proof (continued)

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