Stat 709: Mathematical Statistics
Lecture 1

Jun Shao

Department of Statistics
University of Wisconsin
Madison, WI 53706, USA

Random experiment: uncertainty in outcomes

Sample space

Ω: sample (or outcome) space: a set containing all possible outcomes

Definition 1.1

Let $\mathscr{F}$ be a collection of subsets of a sample space Ω.
$\mathscr{F}$ is called a σ-field (or σ-algebra) iff it has the following properties.
(i) The empty set $\emptyset \in \mathscr{F}$.
(ii) If $A \in \mathscr{F}$, then the complement $A^c \in \mathscr{F}$.
(iii) If $A_i \in \mathscr{F}$, $i = 1, 2, ...$, then their union $\cup A_i \in \mathscr{F}$.
$(\Omega, \mathscr{F})$ is a measurable space if $\mathscr{F}$ is a σ-field on Ω

Discussion

$\mathscr{F}$ is a collection (set) of sets
Two trivial examples: $\mathscr{F}$ contains $\emptyset$ and Ω only;
$\mathscr{F}$ contains all subsets of Ω (power set)

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Why do we need to consider other σ-field?

We may be interested in a particular collection of sets

e.g., $\mathscr{F} = \{\emptyset, A, A^c, \Omega\}$, where $A \subset \Omega$

$\mathscr{C}$ = a collection of sets of interest

$\mathscr{C}$ may not be a σ-field

$\sigma(\mathscr{C})$: the smallest σ-field containing $\mathscr{C}$ (the σ-field generated by $\mathscr{C}$)

$\sigma(\mathscr{C}) = \mathscr{C}$ if $\mathscr{C}$ itself is a σ-field

$\Gamma = \{\mathscr{F} : \mathscr{F}$ is a σ-field on Ω and $\mathscr{C} \subset \mathscr{F}\}$

$\sigma(\mathscr{C}) = \cap_{\mathscr{F} \in \Gamma} \mathscr{F}$

$\sigma(\{A\}) = \sigma(\{A, A^c\}) = \sigma(\{A, \Omega\}) = \sigma(\{A, \emptyset\}) = \{\emptyset, A, A^c, \Omega\}$

Borel σ-field

$\mathscr{R}^k$: the k-dimensional Euclidean space ($\mathscr{R}^1 = \mathscr{R}$ is the real line)

$\mathscr{O}$: the collection of all open sets

$\mathscr{B}^k = \sigma(\mathscr{O})$: the Borel σ-field on $\mathscr{R}^k$

$\mathscr{B}^k = \sigma(\mathscr{C})$, $\mathscr{C}$ is the collection of all closed sets

$C \in \mathscr{B}^k$, $\mathscr{B}_C = \{C \cap B : B \in \mathscr{B}^k\}$ is the Borel σ-field on C

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Measure

Length, area, volume, ...

Definition 1.2.

Let $(\Omega, \mathscr{F})$ be a measurable space.
A set function ν defined on $\mathscr{F}$ is called a *measure* iff it has the following properties.
(i) $0 \leq \nu(A) \leq \infty$ for any $A \in \mathscr{F}$.
(ii) $\nu(\emptyset) = 0$.
(iii) If $A_i \in \mathscr{F}$, $i = 1, 2, \ldots$, and A_i's are disjoint, i.e., $A_i \cap A_j = \emptyset$ for any $i \neq j$, then

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \nu(A_i).$$

$(\Omega, \mathscr{F}, \nu)$ is a measure space if ν is a measure on a σ-field on Ω.

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Conventions

- For any $x \in \mathscr{R}$, $\infty + x = \infty$, $x\infty = \infty$ if $x > 0$, $x\infty = -\infty$ if $x < 0$, and $0\infty = 0$;
- $\infty + \infty = \infty$;
- $\infty^a = \infty$ for any $a > 0$;
- $\infty - \infty$ or ∞/∞ is not defined

Probability measure

If $\nu(\Omega) = 1$, then ν is a probability measure.
We usually use notation P instead of ν.
$(\Omega, \mathscr{F}, P)$ is a probability space if P is a probability measure on a σ-field on Ω.

Important examples of measures

- Measures take ∞ as its value:

$$\nu(A) = \begin{cases} \infty & A \in \mathscr{F}, A \neq \emptyset \\ 0 & A = \emptyset. \end{cases}$$

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Important examples of measures

- Point mass:
  Let $x \in \Omega$ be a fixed point.

$$\delta_x(A) = \begin{cases} c & x \in A \\ 0 & x \notin A. \end{cases}$$

- Counting measure:
  Let Ω be a sample space, $\mathscr{F}$ the collection of all subsets, and $\nu(A)$ the number of elements in $A \in \mathscr{F}$ ($\nu(A) = \infty$ if A contains infinitely many elements).
  Then ν is a measure on $\mathscr{F}$ and is called the *counting measure*.
- Lebesgue measure:
  There is a unique measure m on $(\mathscr{R}, \mathscr{B})$ that satisfies
  $m([a,b]) = b - a$ for every finite interval $[a,b]$, $-\infty < a \le b < \infty$.
  This is called the *Lebesgue measure*.
  If we restrict m to the measurable space $([0,1], \mathscr{B}_{[0,1]})$, then m is a probability measure.

Proposition 1.1 (Properties of measures)

Let $(\Omega, \mathscr{F}, \nu)$ be a measure space.

1. (Monotonicity). If $A \subset B$, then $\nu(A) \leq \nu(B)$.
2. (Subadditivity). For any sequence $A_1, A_2, \ldots,$

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \nu(A_i).$$

3. (Continuity). If $A_1 \subset A_2 \subset A_3 \subset \cdots$ (or $A_1 \supset A_2 \supset A_3 \supset \cdots$ and $\nu(A_1) < \infty$), then

$$\nu\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \nu(A_n),$$

where

$$\lim_{n\to\infty} A_n = \bigcup_{i=1}^{\infty} A_i \quad \left(\text{or } = \bigcap_{i=1}^{\infty} A_i\right).$$

Cumulative distribution function)

Let P be a probability measure on $(\mathscr{R},\mathscr{B})$.
The *cumulative distribution function* (c.d.f.) of P is defined to be

$$F(x) = P((-\infty, x]), \quad x \in \mathscr{R}$$

Proposition 1.2 (Properties of c.d.f.'s)

(i) Let F be a c.d.f. on $\mathscr{R}$.

(a) $F(-\infty) = \lim_{x\to-\infty} F(x) = 0$;

(b) $F(\infty) = \lim_{x\to\infty} F(x) = 1$;

(c) F is nondecreasing, i.e., $F(x) \leq F(y)$ if $x \leq y$;

(d) F is right continuous, i.e., $\lim_{y\to x, y>x} F(y) = F(x)$.

(ii) Suppose a real-valued function F on $\mathscr{R}$ satisfies (a)-(d) in part (i). Then F is the c.d.f. of a unique probability measure on $(\mathscr{R},\mathscr{B})$.

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