DISCUSSION 7

1 Review

- **Binomial Distribution**
  Let the discrete random variable $X$ = “# Successes in $n$ independent Bernoulli trials (0,1,2,...,n),” each having constant probability $P($Success$)=\pi$, and hence $P($Failure$)=1-\pi$. Then the probability of obtaining any specific number of successes $x = 0, 1, 2, ..., n$, is given by:

$$P(X = x) = \binom{n}{x} \pi^x (1 - \pi)^{n-x}.$$  

Then $X$ is a Binomial Distribution, denoted by $X \sim Bin(n, \pi)$.
Furthermore, the mean $\mu = n\pi$, and the standard deviation $\sigma = \sqrt{n\pi(1 - \pi)}$.

- **Poisson Distribution**
  Let the discrete random variable $X$ = “# occurrences of a (rare) event $E$, in a given interval of time of space, of size $T$.” Assume:

  1. All the occurrences of $E$ are independent in the interval.
  2. The mean number $\mu$ of expected occurrences of $E$ in the interval is proportional to $T$, i.e., $\mu = \alpha T$. This constant of proportionality $\alpha$ is called the rate of the resulting Poisson process.

Then the probability of obtaining any specific number $x = 0, 1, 2, ...$ of occurrences of event $E$ is given by:

$$P(X = x) = \frac{e^{-\mu} \mu^x}{x!}.$$  

We say that $X$ has a Poisson Distribution, denoted by $X \sim Poisson(\mu)$.
Furthermore, the mean $\mu = \alpha T$, and the standard deviation $\sigma^2 = \alpha T$.

- **Continuous Random Variable**
  $f(x)$ is a probability density function for the continuous random variable $X$ if, for all $x$,

$$f(x) \geq 0 \quad AND \quad \int_{-\infty}^{\infty} f(x)dx = 1.$$  

The cumulative distribution function (cdf) is defined as, for all $x$,

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt.$$  

Therefore, $F$ increases monotonically and continuously from 0 to 1. Furthermore, $P(a \leq X \leq b) = \int_{a}^{b} f(x)dx = F(b) - F(a)$.

2 Problems
4-5. Compare this problem with 2-10!

Consider the binary population variable \( Y = \begin{cases} 1, & \text{with probability } \pi \\ 0, & \text{with probability } 1 - \pi \end{cases} \) (see figure).

(a) Construct a probability table for this random variable.

(b) Show that the population mean \( \mu_Y = \pi \).

(c) Show that the population variance \( \sigma_Y^2 = \pi (1 - \pi) \).

Note that \( \pi \) controls both the mean and the variance!

4-15. Suppose that the continuous random variable \( X = \) “age of juniors at the UW-Iwanagoeeches campus” is symmetrically distributed about its mean, but piecewise linear as illustrated, rather than being a normally distributed bell curve.

For an individual selected at random from this population, calculate each of the following.

(a) Verify by direct computation that \( P(18 \leq X \leq 22) = 1 \), as it should be.
   \[ \text{[Hint: Recall that the area of a triangle } = \frac{1}{2} \text{ (base } \times \text{ height).]} \]

(b) \( P(18 \leq X < 18.5) \)

(c) \( P(18.5 < X \leq 19) \)

(d) \( P(19.5 < X < 20.5) \)

(e) What symmetric interval about the mean contains exactly half the population values? Express in terms of years and months.

(f) Determine the equation of the density function \( f(x) \), and the cumulative function \( F(x) \). Sketch the graph of \( F(x) \). \[ \text{[Note: This problem can be done without calculus, but it helps.]} \]