Supplement C: Almost Sure Consistency of $p(\gamma^0|Z)$. The consistency results in Theorems 2.2–2.4 hold in probability. We can also establish almost sure consistency. Before stating the exact results, we need to introduce the following technical condition which plays a role similar to that of Assumption 2.3, but imposes a greater restriction on ψ_n.

ASSUMPTION 5.1. *There exists a positive sequence ψ_n such that $\min_{j\in\gamma^0}|\beta_j^0| \geq \psi_n$ and, as $n \to \infty$, $\psi_n\sqrt{n/\log n} \to \infty$.*

With this assumption, we can derive the following results on almost sure convergence when the true model is either nonnull or null.

THEOREM 5.4. *Suppose γ^0 is nonnull, and Assumptions 2.1, 2.2, 2.4, 2.6–2.8 and 5.1 are satisfied. Let $\delta \geq 0$ satisfy Assumption 2.8. If $n^{1-\delta-\alpha_0}\underline{\phi}_n \to \infty$ for some $\alpha_0 > 4$, then $\max_{\gamma\neq\gamma^0} p(\gamma|Z)/p(\gamma^0|Z) \to 0$, a.s. If $p^2 = o(n^{1-\delta-\alpha_0}\underline{\phi}_n)$ for some $\alpha_0 > 4$, then $\sum_{\gamma\neq\gamma^0} p(\gamma|Z) \to 0$, a.s., and consequently, $p(\gamma^0|Z) \to 1$, a.s.*

THEOREM 5.5. *Suppose γ^0 is null. Suppose that Assumptions 2.1, 2.5 and 2.8 are satisfied. If $n^{1-\delta-\alpha_0}\underline{\phi}_n \to \infty$ for some $\alpha_0 > 4$, then $\max_{\gamma\neq\gamma^0} p(\gamma|Z)/p(\gamma^0|Z) \to 0$, a.s. If $p^2 = o(n^{1-\delta-\alpha_0}\underline{\phi}_n)$ for some $\alpha_0 > 4$, then $\sum_{\gamma\neq\gamma^0} p(\gamma|Z) \to 0$, a.s., and consequently, $p(\gamma^0|Z) \to 1$, a.s.*

The following lemma is useful to establish the almost sure convergence of $p(\gamma^0|Z)$.

LEMMA 5.6. *Let $\epsilon \sim N(0, \sigma_0^2 I_p)$.*

(a). Let $v_\gamma = (I_n - P_\gamma)X_{\gamma^0\setminus\gamma}\beta^0_{\gamma^0\setminus\gamma}$. Then $\limsup_{n\to\infty}\max_{\gamma\in S_2}|v_\gamma'\epsilon|/(\|v_\gamma\|_2\sqrt{p\log n}) < 2\sigma_0$, a.s.
(b). If S_1 is nonnull, then $\limsup_{n\to\infty}\max_{\gamma\in S_1}\epsilon^T(P_\gamma - P_{\gamma^0})\epsilon/((|\gamma| - s_n)\log n) \leq 4\sigma_0^2$, a.s.
(c). If S_2 is nonnull, then $\limsup_{n\to\infty}\max_{\gamma\in S_2}\epsilon^T P_\gamma\epsilon/(|\gamma|\log n) \leq 4\sigma_0^2$, a.s.
(d). $\limsup_n \epsilon^T P_{\gamma^0}\epsilon/(s_n\log n) \leq 4\sigma_0^2$, a.s.

Proof of Lemma 5.6. The proofs can be completed by using the Borel-Cantelli lemma and the techniques in the proof Lemma 5.1.

(a) Similar to the proof of part (a) in Lemma 5.1, it can be shown that

$$pr\left(\max_{\gamma\in S_2}\frac{|v_\gamma'\epsilon|}{\|v_\gamma\|_2}\geq 2\sigma_0\sqrt{p\log n}\right)\leq C_0\left(\frac{2}{n^2}\right)^p,$$

therefore, $\sum_n pr\left(\max_{\gamma\in S_2}|v_\gamma'\epsilon|/\|v_\gamma\|_2\geq 2\sigma_0\sqrt{2p\log n}\right)<\infty$. By the Borel-Cantelli lemma, the desired result holds.

(b) For any $\alpha>4$, we temporarily fix $\alpha'>2$ such that $2\alpha'<\alpha$. Following the proof of part (b) in Lemma 5.1, for large n,

$$\begin{aligned} pr\left(\max_{\gamma\in S_1}\frac{\epsilon^T(P_\gamma-P_{\gamma^0})\epsilon}{(|\gamma|-s_n)\log n}\geq\alpha\sigma_0^2\right) &\leq \left(1+(1-2/\alpha')^{-1/2}n^{-\alpha/\alpha'}\right)^{p-s_n}-1\\ &\leq \exp\left((1-2/\alpha')^{-1/2}n^{-\alpha/\alpha'}p\right)-1\\ &\leq 2(1-2/\alpha')^{-1/2}n^{1-\alpha/\alpha'}. \end{aligned}$$

Therefore, by the Borel-Cantelli lemma, $\limsup_n\max_{\gamma\in S_1}\epsilon^T(P_\gamma-P_{\gamma^0})\epsilon/((|\gamma|-s_n)\log n)\leq\alpha\sigma_0^2$, a.s. Then the desired result holds by selecting a sequence of αs approaching 4.

(c) & (d) These proofs can be accomplished by arguments similar to part (b). □

Proof of Theorem 5.4. We start with (5.1) to approximate the ratio $p(\gamma|Z)/p(\gamma^0|Z)$. The terms T_1 and T_3 in (5.1) are bounded below. The approximation of T_2 is given by Lemma 5.2. By Lemma 5.6 (d), Assumptions 2.2, 2.7 and 5.1, and a similar argument to (5.3), it can be shown that $0\leq -T_4=O(1)$, a.s.

To approximate T_5, by Assumption 2.4, Lemma 5.6 and a careful revision of (5.5), (5.6) and (5.7), one can show that with probability equal to one, for some constant $C''>0$, there exists a large N such that if $n\geq N$, uniformly for $\gamma\in S_2$, $T_5\geq 2^{-1}(n+\nu)\log(1+C''\psi_n^2)$; uniformly for $\gamma\in S_1$, $T_5\geq -2^{-1}(|\gamma|-s_n)\alpha_0\log n$.

Thus, by an argument similar to (5.9) and (5.11), it can be shown that $\sum_{\gamma\in S_2}p(\gamma|Z)/p(\gamma^0|Z)\to 0$, a.s., which implies that $\max_{\gamma\in S_2}p(\gamma|Z)/p(\gamma^0|Z)\to 0$, a.s.

On the other hand, by the above approximations of T_1 to T_5, with probability equal to one, there exists

a constant $C''' > 0$ such that when n is sufficiently large, uniformly for $\gamma \in S_1$,

$$\begin{aligned} p(\gamma|Z)/p(\gamma^0|Z) &\leq C''' \exp\left(-2^{-1}(|\gamma| - s_n)\log(1 + C_3 n^{1-\delta}\underline{\phi}_n) + 2^{-1}(|\gamma| - s_n)\log n^{\alpha_0}\right) \\ &= C'''\left(\frac{1 + C_3 n^{1-\delta}\underline{\phi}_n}{n^{\alpha_0}}\right)^{-2^{-1}(|\gamma|-s_n)}. \end{aligned} \tag{5.17}$$

Thus, if $n^{1-\delta-\alpha_0}\underline{\phi}_n \to \infty$, then $\max_{\gamma \in S_1} p(\gamma|Z)/p(\gamma^0|Z) \to 0$, a.s.

By (5.17), with probability equal to one,

$$\begin{aligned} \sum_{\gamma \in S_1} p(\gamma|Z)/p(\gamma^0|Z) &\leq C''' \sum_{\gamma \in S_1}\left(\frac{1 + C_3 n^{1-\delta}\underline{\phi}_n}{n^{\alpha_0}}\right)^{-2^{-1}(|\gamma|-s_n)} \\ &= C'''\left[\left(1 + \left(\frac{1 + C_3 n^{1-\delta}\underline{\phi}_n}{n^{\alpha_0}}\right)^{-1/2}\right)^{p-s_n} - 1\right]. \end{aligned}$$

Thus, if $p^2 = o(n^{1-\delta-\alpha_0}\underline{\phi}_n)$, then $\sum_{\gamma \in S_1} p(\gamma|Z)/p(\gamma^0|Z) \to 0$, a.s., which, by (2.7), implies $p(\gamma^0|Z) \to 1$, a.s. □

Proof of Theorem 5.5. Proof is completed by similar arguments to the proofs of Theorems 2.4 and 5.4. □