Statistics 701 Assignment 1 Due Sept. 22, 2003

1. Suppose ε_t are independent identically distributed (i.i.d.) r.v.'s with mean 0 and variance σ^2, and define a stochastic process $\{Y_t\}$ by $Y_t = \mu + \varepsilon_t - (2/3)\,\varepsilon_{t-1}$, $t = \ldots, -1, 0, 1, 2, \ldots$, where μ is a constant. Find the mean, autocovariance function, and ACF of Y_t and verify that $\{Y_t\}$ is a stationary process. Do the same thing for the process defined by $Y_t = \mu + a_t - (3/2)\,a_{t-1}$, where the a_t are i.i.d. with mean 0 and variance σ_a^2, and compare the ACF's of the two processes.
   Also, give an explicit expression for the spectral density function $f(\lambda)$, $0 \le \lambda \le \pi$, for the first process, make a rough plot of $f(\lambda)$ versus λ, and briefly interpret.

2. Let $\{Y_t\}$ be a stationary process with mean $\mu_y = E(Y_t)$ and autocovariance function $\gamma_y(s) = \mathrm{Cov}(Y_t, Y_{t+s})$, $s = 0, \pm 1, \pm 2, \ldots$. A new process (first differences) is defined by $W_t = Y_t - Y_{t-1}$, $t = \ldots, -1, 0, 1, \ldots$. Obtain the mean of $\{W_t\}$ and the autocovariance function of $\{W_t\}$ in terms of $\gamma_y(s)$, and show that $\{W_t\}$ is a stationary process.
   Also, find the autocorrelation function $\rho_w(s)$ of $\{W_t\}$ when the autocorrelation function of $\{Y_t\}$ is $\rho_y(s) = \phi^{|s|}$, where ϕ is a constant, $|\phi| < 1$; evaluate the ACF $\rho_w(s)$ for the first 5 lags $s = 1, \ldots, 5$ when $\phi = 0.8$.

3. Suppose $\{Y_t\}$ is a process defined by $Y_t = \beta_0 + \beta_1 t + X_t$, where $\{X_t\}$ is a stationary process with autocovariance function $\gamma_x(s) = \mathrm{Cov}(X_t, X_{t+s})$, and β_0 and β_1 are constants. State the mean function and the autocovariance function for $\{Y_t\}$, deduce that $\{Y_t\}$ is not a stationary process, and show that $W_t = Y_t - Y_{t-1}$ is stationary.
   [Note: For the last part, you can use results established in Problem 2.]

4. Suppose $\{Y_t\}$ and $\{Z_t\}$ are two stationary processes with autocovariance functions $\gamma_y(s)$ and $\gamma_z(s)$, such that $\{Y_t\}$ and $\{Z_t\}$ are completely independent, i.e., Y_{t_1} and Z_{t_2} are independent r.v.'s for all t_1 and t_2. Show that the process defined by $W_t = a\,Y_t + b\,Z_t$, where a, b are constants, is stationary, by determining the mean and autocovariance function of $\{W_t\}$.

5. (a) Consider the process $\{Y_t\}$ defined by $Y_t = \varepsilon_1 \cos(\lambda t) + \varepsilon_2 \sin(\lambda t)$, $t = \ldots, 0, 1, \ldots$, where ε_1 and ε_2 are two i.i.d. r.v.'s with mean 0 and variance σ^2, and λ is a constant. Determine the mean and autocovariance function of $\{Y_t\}$ to show that $\{Y_t\}$ is a stationary process. How will a sample realization of $\{Y_t\}$ "behave"? [Hint: Is the process $\{Y_t\}$ deterministic ?]
   (b) Define the process $\{Y_t\}$ by $Y_t = \sum_{i=1}^{K} [\,\varepsilon_{1i} \cos(\lambda_i t) + \varepsilon_{2i} \sin(\lambda_i t)\,]$, where the λ_i are constants and the $\varepsilon_{11}, \ldots, \varepsilon_{1K}, \varepsilon_{21}, \ldots, \varepsilon_{2K}$ are independent r.v.'s with mean 0 and variances $\sigma_i^2 = \mathrm{Var}(\varepsilon_{1i}) = \mathrm{Var}(\varepsilon_{2i})$, $i = 1, \ldots, K$. Compute the mean and autocovariance function of $\{Y_t\}$ to show that $\{Y_t\}$ is stationary.
   [Hint: In problem 5, use the trigonometric identity $\cos(x)\cos(y) + \sin(x)\sin(y) = \cos(x - y)$.]

6. Let $\{\varepsilon_t\}$, $t = 1, 2, \ldots$, be i.i.d. r.v.'s with mean 0 and variance σ^2, and define the process $\{Y_t\}$ by

$$Y_t = \phi^t\, Y_0 + \sum_{i=0}^{t-1} \phi^i\, \varepsilon_{t-i} = \phi^t\, Y_0 + \varepsilon_t + \phi\, \varepsilon_{t-1} + \phi^2\, \varepsilon_{t-2} + \cdots + \phi^{t-1}\, \varepsilon_1\,, \quad t = 1, 2, \ldots,$$

with $Y_0 = 0$ and where ϕ is a constant, $|\phi| < 1$. Compute the autocovariance $\mathrm{Cov}(Y_t, Y_{t+s})$ for $s > 0$ and any $t = 1, 2, \ldots$, and show that $\{Y_t\}$ is not a stationary process. However, verify that for "large" values of t (i.e., as $t \to \infty$) the dependence of $\mathrm{Cov}(Y_t, Y_{t+s})$ on t becomes negligible, using the fact that $\sum_{i=0}^{t-1} a^i = (1 - a^t)/(1 - a)$ and $a^t \to 0$ as $t \to \infty$ for any a such that $|a| < 1$. Thus we might say that the process $\{Y_t\}$ is "approximately stationary" for large t. Also, show that the process $\{Y_t\}$ can be expressed in the recursive form as

$$Y_t = \phi\, Y_{t-1} + \varepsilon_t\,, \quad t = 1, 2, 3, \ldots, \quad Y_0 = 0\,.$$

If we alter the above definition of $\{Y_t\}$ by assuming that Y_0 is a zero-mean r.v. independent of the ε_t, $t > 0$, with $\mathrm{Var}(Y_0) = \sigma^2/(1 - \phi^2)$, then $\{Y_t\}$ will be a stationary process. Verify this!
[Comment: The above results have implications for computer simulation of a stationary first-order autoregressive process.]

7. Problem 2.1 (of exercises for Chapter 2) in Box, Jenkins, and Reinsel. Also,
(iv) After inspecting the graphs (i)-(iii), comment on whether you think the series is stationary.
(v) Calculate the sample autocorrelations $r(k)$, $k = 1, 2, \ldots, 6$ for this series. Make a graph of the $r(k)$, $k = 1, 2, \ldots, 6$. How do the values of $r(1)$ and $r(2)$ relate to your graphs in (ii) and (iii) ?
(vi) On the assumption that $\rho(k) = 0$ for $k > 2$, obtain approximate standard errors for $r(1)$, $r(2)$, and $r(k)$ for $k > 2$, and also obtain the approximate correlation between $r(4)$ and $r(5)$.

8. Let $Y_1, Y_2, \ldots, Y_T, \ldots$ form a stationary stochastic process with mean $\mu = E(Y_t)$, variance $\gamma(0) = \mathrm{Var}(Y_t)$, and autocorrelation function $\rho(s) = \gamma(s)/\gamma(0)$, where $\gamma(s) = \mathrm{Cov}(Y_t, Y_{t+s})$. Show that if $\rho(s) \to 0$ as $s \to \infty$ then the sample mean $\bar{Y} = (1/T) \sum_{t=1}^{T} Y_t$ converges in probability to μ as $T \to \infty$ (that is, $Pr\{|\bar{Y} - \mu| \geq a\} \to 0$ as $T \to \infty$ for every $a > 0$).
[Hint: Establish that

$$\mathrm{Var}(\bar{Y}) = \frac{\gamma(0)}{T} + \frac{2}{T^2} \sum_{s=1}^{T-1} (T - s)\, \gamma(s)\,,$$

and from this determine that $\mathrm{Var}(\bar{Y}) \to 0$ as $T \to \infty$ if $\gamma(s) \to 0$ as $s \to \infty$. Then apply Chebyshev's inequality, i.e., $Pr\{|X - E(X)| \geq a\} \leq \mathrm{Var}(X)/a^2$, for any $a > 0$.]

Also, compare $\mathrm{Var}(\bar{Y})$ in the case where the Y_t are i.i.d. r.v.'s with $\mathrm{Var}(\bar{Y})$ in the case where the Y_t have autocorrelation function $\rho(1) = 0.6$, $\rho(2) = 0.3$, $\rho(s) = 0$ for $s > 2$.