Stat 992: Lecture 36
Smoothness of random fields

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Solution to Problem 37. (Tulaya Limpiti). The smoothness of noise can be measured via the covariance matrix of the derivative field.

1. Consider zero mean unit variance Gaussian field $Y = K_\sigma * W$ where $W$ Gaussian white noise with the Dirac-delta covariance function. The covariance of $Y$ has been shown to be

$$R_Y(x, y) = \int K_\sigma(x-z)K_\sigma(y-z)\,dz$$

which should be isotropic, i.e. $R_Y(x, y) = f(\|x-y\|^2)$ for some function $f$. Field $Y$ does not give unit variance. So we normalize $Y$ by the square root of

$$\text{Var} Y = R_Y(x, x) = f(0) = \int K_\sigma^2(x-z)\,dz.$$  

Note $K_\sigma^2(x) = \frac{1}{(2\sqrt{\pi}\sigma)^n}K_\sigma/\sqrt{x}(x)$. So $Y = (2\sqrt{\pi}\sigma)^n/2 K_\sigma * W$ gives a unit variance field. The way to generate this field numerically is we first simulate white noise $N(0, 1)$ and perform kernel smoothing followed by the multiplication of $(2\sqrt{\pi}\sigma)^n/2$.

2. The cross-covariance of the above zero mean unit variance field is given by

$$E[\partial_{x_i} Y(x) \partial_{x_j} Y(y)] = \int \frac{\partial \sigma(x-z)K_\sigma(y-z)}{\partial \sigma(z)K_\sigma(z)}\,dz$$

Then

$$\text{Var}(\partial Y) = \int \frac{\partial K_\sigma(x)[\partial K_\sigma(x)]^T}{\partial K_\sigma(x)K_\sigma(x)}\,dx.$$  

3. For any isotropic fields $Y_i$ it can be shown that $\partial_{x_i} Y$ and $\partial_{x_j} Y$ are uncorrelated if $i \neq j$. It is a trivial property of the isotropic field. We have cross-covariance

$$E[\partial_{x_i} Y(x) \partial_{x_j} Y(y)] = \partial_{x_i} \partial_{x_j} f(\|x - y\|^2).$$

Since $f(\|x\|^2) = f(\sum_{i=1}^n x_i^2)$, $\partial_{x_i} f(\|x\|^2) = 2x_i f'(\|x\|^2)$. Then,

$$\partial_{x_i} \partial_{x_j} f(\|x\|^2) = 2\delta_{ij} f'(\|x\|^2) + 4x_i x_j f''(\|x\|^2)$$

Letting $x = y$ in equation (1),

$$E[\partial_{x_i} Y(x) \partial_{x_j} Y(x)] = \partial_{x_i} \partial_{x_j} f(0) = 2\delta_{ij} f'(0).$$

Hence all partial derivatives are uncorrelated.

A different way to see this is to compute the covariance directly from (3).

$$E[\partial_{x_i} Y(x) \partial_{x_j} Y(x)] = \int \frac{\partial \sigma_i(x)[\partial \sigma_j(x)]^T}{\partial \sigma(x)\sigma(x)}\,dx$$

where this formula is given.


Next lecture will continue the discussion of anisotropic smoothing.