1. Suprema distributions. Following previous lectures, for zero mean random fields $Y$, type I-error $\alpha$ was given by
\[
\alpha = P\left( \bigcup_{x \in \Omega} \{Y(x) > h\} \right) = P\left( \sup_{x \in \Omega} Y(x) > h \right).
\]
So we need to find the distribution of a random variable $\sup_{x \in \Omega} Y(x)$. This is a difficult math problem. Let us compute some simple suprema distributions for Gaussian processes.

**Theorem. (reflection principle)** Consider a Brownian motion $B(x)$ in interval $\Omega = [0, u]$ with $P(B(0) = 0) = 1$. Then
\[
P\left( \sup_{0 \leq x \leq u} B(x) > h \right) = \frac{2}{\sqrt{2\pi u}} \int_0^\infty e^{-x^2/2u} \, dx.
\]
**Proof.** Note this proof is not rigorous but intuitive. For a rigorous proof, see Freedman’s Brownian Motion and Diffusion. Let $M_u = \sup_{0 \leq x \leq u} B(x)$. Define hitting time
\[
T_h = \inf\{x : 0 \leq x, B(x) = h\}.
\]
By treating $x$ to be time, $T_h$ is the first time the Brownian motion hits $h$. Note that
\[
P(M_u \geq h) = P(T_h \leq u).
\]
So by knowing the distribution of $T_h$, we know the distribution of $M_u$. Define a reflected process $\tilde{B}(x)$ to be
\[
\tilde{B}(x) = \begin{cases} B(x) & \text{for } x < T_h, \\ h - [B(x) - h] & \text{for } x \geq T_h. \end{cases}
\]
Note that $B$ and $\tilde{B}$ are equal in distribution from symmetry. Then
\[
P(M_u \geq h) = P(M_u \geq h, B(u) \geq h) + P(M_u \geq h, B(u) < h)
\]
\[
= P(B(u) \geq h) + P(\tilde{B}(u) > h)
\]
\[
= 2P(B(u) > h)
\]

2. Smooth Gaussian process. Read R.J. Adler’s On Excursion Sets, Tube Formulae, and Maxima of Random Fields (1999). The suprema of Brownian motion is simple due to its independent increment properties. Now consider 1-dimensional smooth isotropic Gaussian random process $Y(x), x \in \Omega = [0, 1] \subset \mathbb{R}$. Let $N_h$ to be the number of times $Y$ crosses over $h$ from below (upcrossing) in $[0, 1]$. Then
\[
P\left( \sup_{x \in [0,1]} Y(x) > h \right) = P(N_h \geq 1 \text{ or } Y(0) > h)
\]
\[
\leq P(N_h \geq 1) + P(Y(0) > h)
\]
\[
\leq \mathbb{E}N_h + P(Y(0) > h).
\]
Let $\sigma^2 = \mathbb{E}Y^2(x)$. It can be shown that from Rice formula (1945),
\[
\mathbb{E}N_h = C_1 \exp(h^2/2\sigma^2)
\]
for some constant $C_1$ given in Adler (1999). Also note that $P(Y(0) > h) = 1 - \Phi(h/\sigma)$ where $\Phi$ is the cumulative distribution function of the standard normal. Then
From the well known inequality (Feller, 1968, p.193),
\[
(1 - \frac{\sigma^2}{h^2}) \frac{\sigma}{\sqrt{2\pi h}} e^{-h^2/2\sigma^2} \leq 1 - \Phi\left( \frac{h}{\sigma} \right) \leq \frac{\sigma}{\sqrt{2\pi h}} e^{-h^2/2\sigma^2}
\]
So
\[
P\left( \sup_{x \in [0,1]} Y(x) > h \right) \leq C_1 + \frac{C_2}{h} e^{h^2/2\sigma^2}.
\]
In fact we can show that
\[
P\left( \sup_{x \in [0,1]} Y(x) > h \right) = \left[C_1 + \frac{C_2}{h} + O(h^{-2})\right] e^{h^2/2\sigma^2}.
\]

**Problem 36.** Simulate 10,000 smooth zero mean isotropic Gaussian fields in $[0, 1]$ in MATLAB and check the validity Rice formula by the Monte-Carlo method. You need to plot $\mathbb{E}N_h$ as a function of $h$. Also plot $P\left( \sup_{x \in [0,1]} Y(x) > h \right)$ as a function of $h$ based on your simulation and check the above asymptotic result is correct.