1. Problem statement. In the previous lecture we showed how to estimate Laplacian in regular grid $\mathbb{Z}^n$. Now we show how to estimate Laplacian in irregular grid such as polygonal surfaces in $\mathbb{R}^2$ for the formulation of diffusion smoothing (Figure 1). The question is how one estimate Laplacian or any other differential operators in the polygonal surface. Assume we have observations $Y_i$ at each point $p_i$, which is assumed to follow additive model

$$Y_i = \mu(p_i) + \epsilon(p_i), p_i \in \mathbb{R}^2$$

where $\mu$ is a smooth continuous function and $\epsilon$ zero mean Gaussian random fields. We want to estimate

$$\Delta \mu(p_0) = \left. \frac{\partial^2 \mu}{\partial x^2} \right|_{p_0} + \left. \frac{\partial^2 \mu}{\partial y^2} \right|_{p_0}.$$ 

Unfortunately, the geometry of polygonal surfaces forbid direct application of finite difference scheme. There are a couple of radically different approaches answering this problem (See one of my publication that uses finite element method).

2. Local polynomial regression. Let $p_i = (x_i, y_i)$ be the coordinate of vertices of polygons. We estimate the Laplacian at $p_0 = (0, 0)$ by fitting a quadratic polynomial of the form

$$\mu(u, v) = \beta_0 + \beta_1 u + \beta_2 v + \beta_3 u^2 + \beta_4 uv + \beta_5 v^2.$$  

Let $(u, v) = (x_i, y_i)$ and $\mu(x_i, y_i) = Y_i$ and set up a normal equation, i.e.,

$$Y_i = \beta_0 + \beta_1 x_i + \beta_2 y_i + \beta_3 x_i^2 + \beta_4 x_i y_i + \beta_5 y_i^2.$$ 

Let $Y = (Y_1, \cdots, Y_m)'$, $\beta = (\beta_0, \cdots, \beta_5)'$ and design matrix

$$X = \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 \\ 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_m & y_m & x_m^2 & x_m y_m & y_m^2 \end{pmatrix}.$$

Then we have the following matrix equation

$$Y = X \beta.$$

The unknown coefficients $\beta_i$ are estimated by the least-squares approximation such that

$$\hat{\beta} = (\hat{\beta}_0, \ldots, \hat{\beta}_5)' = (X'X)^{-1}X'Y,$$

where $^{-1}$ denotes generalized inverse, which can be obtained through the singular value decomposition (SVD). Note that $X'X$ is nonsingular if $m > 6$.


$$XX^-X = X, X^-XX^- = X^-,$$

$$(XX^-)' = XX^-, (X^-X)' = X^-X.$$ 

Let $X$ be $m \times p$ matrix with $m \geq p$. Then SVD of $X$ is

$$X = UDV',$$

where $U_{m \times p}$ has orthonormal columns, $D_{p \times p} = \text{Diag}(d_1, \cdots, d_p)$ is diagonal with non-negative
elements and $V_{p \times p}$ is orthogonal. Let $D^{-} = \text{Diag}(d_{1}^{-}, \cdots, d_{p}^{-})$, $d_{i}^{-} = 1/d_{i}$ if $d_{i} \neq 0$ and $d_{i}^{-} = 0$ if $d_{i} = 0$. Then it can be shown that the Moore-Penrose generalized inverse is given by

$$X^{-} = V D^{-} U'.$$

Then the Laplacian of is

$$\hat{\Delta} \mu(p_{0}) = 2\hat{\beta}_{3} + 2\hat{\beta}_{5}.$$

Note that $\hat{\beta}_{3}$ and $\hat{\beta}_{5}$ are expressed as the weighted averaging of $Y_{i}$'s. Hence

$$\hat{\Delta} \mu(p_{0}) = \sum_{i=1}^{m} w_{i} Y_{i}$$

where weight $w_{i}$ are equivalent to the the sum of the $i$-th component of the 3rd and 5th rows of $X^{-}$. Note that the weights $w_{i}$ are functions of the coordinates of vertices $p_{1}, \cdots, p_{m}$.

**Problem 28.** Express weights $w_{i}$ in terms of $x_{i}$'s and $y_{i}$'s. **MAPLE** or **MATHEMATICA** may help you. :)