Problem 4. Investigate the property of the kernel estimator $K_H * Y$ when $H \to \infty$. There are a couple of ways to prove what happens when $H \to \infty$. It will be shown later that the estimator is a solution to a heat equation and the condition $H \to \infty$ is equivalent to the steady state when we diffuse heat for infinite amount of time.

Solution. We will show it for 1D isotropic case. General case follows similarly. Note that $K_\sigma \leq \frac{1}{\sqrt{2\pi}\sigma}$. So

$$|K_\sigma * Y(x)| \leq \frac{1}{\sqrt{2\pi}\sigma} \left| \int Y(x) \, dx \right|.$$  

As $\sigma \to \infty$, it converges to 0. So it seems the kernel smoothing is completely useless when the bandwidth is extremely large. However things change dramatically if we use truncated kernel $\tilde{K}_\sigma$. Assume $Y$ is defined in finite domain $\Omega$. We truncate and normalize kernel such that

$$\tilde{K}_\sigma = \frac{K_\sigma(x)1_\Omega(x)}{\int_\Omega K_\sigma(x) \, dx} = \frac{e^{-x^2/2\sigma^2}}{\int_\Omega e^{-x^2/2\sigma^2} \, dx} 1_\Omega(x)$$

where $1_\Omega(x)$ is an indicator function. Then trivially

$$\lim_{\sigma \to \infty} \tilde{K}_\sigma = \frac{1_\Omega(x)}{\int_\Omega \, dx}.$$  

Hence

$$\lim_{\sigma \to \infty} \tilde{K}_\sigma * Y(x) = \frac{\int_\Omega Y(x) \, dx}{\int_\Omega \, dx}$$

which is the average signal over $\Omega$.

The kernel smoothing corresponds to the solution of a heat equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$

with initial condition $f(x, 0) = Y(x)$ at time $t = 2\sigma^2$. As $\sigma \to \infty, t \to \infty$. Heat diffusion after infinite amount of time corresponds to steady state $\frac{\partial f}{\partial t} = 0$ and the same conclusion can be drawn.

Problem 5.

$$\inf \mu(t) \leq \mathbb{E}\hat{\mu}(t) \leq \sup \mu(t).$$  \hspace{1cm} (1)  

Inequality (1) implies that the smoothed signal will be smaller than the maximum and larger than the minimum of the signal. Can you come up with a sharper bound than (1)? Hint: assume $\mu$ has a finite support and work on the neighborhood of $t$.

Solution. We will prove it for isotropic case but anisotropic case is similar. We need to assume $|\mu| < \infty$ and $\mu$ to have compact support. Let us partition $\mathbb{R}^n = \bigcup_{j=1}^m \Omega_j$ be a partition. Then

$$\mathbb{E}\hat{\mu}(t) = \sum_{j=1}^m \int_{\Omega_j} K_\sigma(t - s) \mu(s) \, ds$$

$$\leq \sum_{j=1}^m \epsilon_j \sup_{s \in \Omega_j} \mu(s),$$

where $\epsilon_j = \int_{\Omega_j} K_\sigma(t - s) \, ds$ and $\sum_{j=1}^m \epsilon_j = 1$. Suppose $\arg \max_{1 \leq j \leq m} \sup_{s \in \Omega_j} \mu(s) = j_0$. We can construct partition such that only single partition $\Omega_{j_0}$ gives the maximum. So the global maximum is obtained in partition $\Omega_{j_0}$, i.e.

$$\sup_{s \in \mathbb{R}^n} \mu(s) = \sup_{s \in \Omega_{j_0}} \mu(s).$$

Then the largest value of $\sum_{j=1}^m \epsilon_j \sup_{s \in \Omega_j} \mu(s)$ is obtained if $\epsilon_{j_0} = 1$ and other $\epsilon_j$’s vanish which do not happen since we can always make $0 < \epsilon_{j_0} < 1$. Hence

$$\sum_{j=1}^m \epsilon_j \sup_{s \in \Omega_j} \mu(s) < \sup_{s \in \mathbb{R}^n} \mu(s).$$

Similarly we can bound by below so that

$$\inf_{s \in \mathbb{R}^n} \mu(s) < \sum_{j=1}^m \epsilon_j \sup_{s \in \Omega_j} \mu(s).$$

There is a degenerate situation this proof breaks down. Consider $\mu(t) = c < \infty$. In that case, we should have a single partition $\Omega_{j_0} = \mathbb{R}^n$ and everything collapses to constant $\inf_{s \in \mathbb{R}^n} \mu(s) = K_\sigma * \mu = \sup_{s \in \mathbb{R}^n} \mu(s) = c$.

Problem 24. Can you come up with a shaper bound than this? There is. :) Hint: For fixed $t$ consider a partition $\Omega_{j_0}$ that contains $t$. 