Stat 992: Lecture 10
Hilbert Space.

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February 11, 2004

1. Hilbert space. In the previous lecture, we showed how to generate Gaussian fields using simple kernel smoothing. A more general technique is based on orthogonal expansions of the fields.

A Hilbert space $\mathcal{H}$ is a complete inner product space. See F. Riesz and B. Sz.-Nagy’s Functional Analysis (1955). For every $f, h \in \mathcal{H}$, there is a scalar number $\langle f, h \rangle$ such that mapping $f \rightarrow \langle f, h \rangle$ is a linear functional, i.e. $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$. Letting $\alpha = 0$ and $\beta = 0$, $(0, h) = 0$. Further we assume $\langle f, g \rangle = \langle g, f \rangle$ and $\langle f, f \rangle > 0$ if $f \neq 0$. We call such mapping $\langle \cdot, \cdot \rangle$ an inner product.

We define metric norm of an element $f$ as $\|f\| = \langle f, f \rangle^{1/2}$. $\mathcal{H}$ is complete if a sequence of elements $f_n \in \mathcal{H}$ with the condition $\|f_n - f_m\| \rightarrow 0$ for $m, n \rightarrow \infty$ has a limit in $\mathcal{H}$.

A space of functions $L^2(0, 1)$ with inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x) \, dx$$

such that $\|f\| < \infty$ is a Hilbert space. This space is usually denoted by $L^2(0, 1)$. The completeness of $L^2(0, 1)$ is a standard exercise in real analysis.

2. Orthogonal expansion. If $U \subset \mathcal{H}$, the span of $U$ denoted by $\text{Span}(U)$ is the collection of all finite linear combinations of elements in $U$. Further define a closure of $\text{Span}(U)$ to be the smallest closed set containing $\text{Span}(U)$. For example consider $H = L^2(0, 1)$. $U = \{1, x, x^2, \cdots \} \subset H$, $\text{Span}(U)$ is the collection of all polynomials while $\overline{\text{Span}(U)} = H$. So in this particular example, $U$ is dense in $H$. The whole idea of orthogonal expansion is to find a relatively smaller countable set $U$ that spans $H$. We call a set separable if it contains a countable spanning subset. So $L^2(0, 1)$ is separable.

Most Hilbert spaces in practice are separable. So for separable Hilbert space $\mathcal{H}$, we have a spanning set $\{u_j\}_{0}^{\infty}$. Let $U = \{u_j\}_{0}^{m}$ be the subset of independent elements. Then using the Gram-Schmidt orthogonalization, we can construct orthonormal basis $V = \{v_j\}_{0}^{m}$ from $U = \{u_j\}_{0}^{m}$ such that

$$v_j = \frac{u_j}{\|u_j\|}, v_j = u_j - \sum_{i=0}^{j-1} \langle u_j, e_i \rangle e_i.$$ 

Then for any element $f \in \mathcal{H}$,

$$f = \sum_{j=0}^{\infty} \alpha_j v_j.$$ 

The coefficient of the expansion is given by $\alpha_j = \langle f, v_j \rangle$ and it is usually termed generalized fourier coefficients for the obvious reason.

We can do a better approximation than this. Based on finite basis $\{v_j\}_{0}^{m}$, we can approximate any element $f \in \mathcal{H}$ in the least-square sense by minimizing

$$\min_{\alpha_j} \left\| f - \sum_{0}^{m} \alpha_j v_j \right\|.$$ 

The minimum always exists in Hilbert space and this is one reason why the least-squares estimation is a widely method. What happen when we use additional basis $v_{m+1}$ to improve the approximation? Nice thing about orthogonal expansion is that there is no need to estimate $\alpha_j$ again since they are given by $\alpha_j = \langle f, v_j \rangle$ for all $j = 0, \ldots, m + 1$.

Problem 20. Construct orthonormal basis $v_j(x), x \in S^1$ that can be used to estimate an integrable function on a unit circle. This can be used in representing functional data on a unit circle, smoothing data via a heat equation etc. After construction, simulate Gaussian random field $\sum_{j=0}^{m} Z_j v_j(x)$ where i.i.d. $Z_j \sim N(0, \sigma^2)$ in computer. What happens when you add additional basis $v_{j+1}$?