Stat 992: Lecture 02
Linear operators on fields.

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January 26, 2004

1. *Differentiation.* The continuity and differentiability of a random fields boil down to the convergence of sequences of random fields. The sequence of random fields $X_h(t)$ converges to $X(t)$ as $h \to 0$ in the mean-square sense if

$$\lim_{h\to 0} \mathbb{E}|X_h(t) - X(t)|^2 = 0.$$

We denote it by $\lim_{h\to 0} X_h(t) = X(t)$ if there is no ambiguity. The convergence in L_2 normal implies the convergence in L_1 norm as we can see in the following arguemnt due to Prof. Tsui.

$$\mathbb{E}|X_h(t)-X(t)|^2 = -\mathbf{Var}\big[X_h(t)-X(t)\big]^2 - \big[\mathbb{E}X_h(t)-X(t)\big]^2.$$

Now let $X_h \to X$ in mean squre. The right hand side is negative and both terms should converge to zero.

The mean-square derivative is defined as

$$\frac{\partial X(t)}{\partial t} = \lim_{h\to 0} \frac{X(t+h) - X(t)}{h}.$$

Partial derivatives are defined similarly. Since differentiation is a linear operator, differentiation of a Gaussian field is again Gaussian with a different covariance function. For zero mean Gaussian fields, the covariance function is $R(t,s) = \mathbb{E}X(t)X(s)$. So the covariance function of the derivative field X' is

$$\mathbb{E}\Big[\frac{dX(t)}{dt}\frac{dX(s)}{ds}\Big] = \frac{\partial^2 R(t,s)}{\partial t \partial s}.$$

Problem 2. Given zero mean Gaussian field $X(t), t = (t_1, \cdots, t_n) \in \mathbb{R}^n$, compute the expectation of the determinant of the Hessian of $X(t)$. Hessian matrix field $H(t)$ is given as $H(t) = \Big(\frac{\partial^2 X(t)}{\partial t_i \partial t_j}\Big)$. Do it for $n = 2$. 12 bonus points for doing it in general n. Hint: for general n, you need to know how to decompose the expectation of the products as the sum of pairwise expectations in Gaussian random variables.

2. *Integration.* Integral is defined as the limit of Riemann sum. Let $\cup_{i=1}^n \Omega_i$ be a partition of $\Omega \subset \mathbb{R}^n$, i.e. $\Omega = \cup_{i=1}^n \Omega_i$ and $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$. Let $t_i \in \Omega_i$ and $\mu(\Omega_i)$ be the volume of Ω_i. Then

$$\int_\Omega X(t)\, dt = \lim_{\text{all } \mu(\Omega_j)\to 0} \sum_{i=1}^n X(t_i)\mu(\Omega_i).$$

Multiple integration is defined similarly. When we integrate a Gaussian field, it is the limit of a linear combination of Gaussian random variables so it is again Gaussian.

Problem 3. Let $\Omega = [0,1]^n \subset \mathbb{R}^n$. Let $X(t)$ be a zero mean Gaussian fields in Ω with covariance function R. Find the distribution of $\int_\Omega X(t)\, dt$.

3. *Linear filter.* We will investigate the linear system of the form

$$Y(t) = \int K(t,s)X(s)\, ds.$$

where K is called the kernel of the integral. This is called the *convolution* of K and X and denoted by

$$Y(t) = K * X(t). \tag{1}$$

In fMRI analysis, for given input $X \in \mathbb{R}^4$ and observation $Y \in \mathbb{R}^4$, estimating arbitrary K in linear system (1) is an important problem. Formulate deconvolution analysis via the finite element method (FEM). Data will be provided if you want to take this as a project. James Ramsay worked on this problem a few years ago.

4. *Dirac-delta.* Formulation (1) is also called kernel smoothing. See Functional Data Analysis by Ramsay and Silverman (1997) for a brief introduction to smoothing in the context of nonparametric regression. We assume the kernel to be isotropic, i.e. $K(t,s) = K(t-s)$ and $\int K(t)\, dt = 1$. Further we may assume K to be unimodal with some parameter σ such that

$$\lim_{\sigma\to 0} K(t;\sigma) \to \delta(t)$$

the Dirac-delta function. Read Quantum Mechanics by Dirac, P.A.M. (1958) firsthand where he introduced the Dirac-delta for the first time. It is a special case of *generalized functions* or *distributions*. For mathematical treatment of distribution theory, read Generalized functions by Gelfand and Shilov (1964) and Green's Functions and Boundary Value Problem by Stakgold (1997). It is defined as

$$\int \delta(t-s)f(s)\,ds = f(t)$$

for any function f. An alternate definition is $\delta(t)=0$ if $t \neq 0$ and $\int \delta(t)\,dt = 1$. This is why it is usually called the *impulse function* in engineering literature.

HOMEWORK 01. Submit solutions to problems 1 to 3 by Feb 2 11:00am. If you don't like these problems, there is no need to submit it. To get full credit, you only need to solve two third of total homework problems. Some problems are intentionally vague.