1. Consider a simple linear model

\[ Y_i = \beta_0 + \epsilon_i, \quad i = 1, \ldots, n \]

where \( \epsilon_i \sim N(0, 1) \). Suppose 3, 2, 4, 2, 1 are observations. We know somehow that the prior to be \( \beta_0 \sim N(3, 1) \). Estimate \( \beta_0 \) using MCMC.

**Solution.** This is a dumb example but serves the purpose of illustrating the use of MCMC for parameter estimation. Note that

\[ Y_i \sim N(0, 1). \]

The posterior is

\[
\pi(\beta_0 | y) = cf(y_1, \ldots, y_n | \beta_0) \pi(\beta_0) \tag{1}
\]

\[
= cf(y_1 | \beta_0) \cdots f(y_n | \beta_0) \pi(\beta_0) \tag{2}
\]

where \( c \) is a normalizing constant, i.e.

\[
c^{-1} = \int_{-\infty}^{\infty} f(y_1, \ldots, y_n | \beta_0) \pi(\beta_0) \, d\beta_0.
\]

Then the posterior is

\[
\pi(\beta_0 | y) \propto \exp \left( -\frac{(\beta_0 - 3)^2}{2} \right) \exp \left( -\frac{\sum_{i=1}^{n}(\beta_0 - y_i)^2}{2} \right).
\]

The Bayes estimate \( \hat{\beta}_0 = \mathbb{E}(\beta_0 | y) \) can be computed analytically in this particular example but requires some algebraic manipulation and if the prior is something like \( \pi(\beta_0) \propto \frac{1}{1 + \beta_0} \), it may not even possible to compute the Bayes estimate analytically.

This gives a motivation for computing \( \hat{\beta}_0 \) via MCMC. We use the random-walk Metropolis algorithm with symmetric propositional density \( q(y | x) \sim N(x, 1) \). For this particular distribution

\[
q(y | x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2}} = q(x | y).
\]

So function \( \alpha \) in the MCMC algorithm is trivially

\[
\alpha(a, b) = \min \left[ 1, \frac{\pi(a | y)}{\pi(b | y)} \right].
\]

Then we generate Markov chain \( X_0 \rightarrow X_1 \rightarrow X_2 \cdots \) with invariant distribution \( \pi(\beta_0 | y) \). For large \( i \), \( X_i \sim \pi(\beta_0 | y) \).

2. **Hierarchical model.** Assume \( Y_i \sim N(\beta_0, 1) \) in the previous model. Give prior \( \beta_0 \sim \pi(\beta_0 | \gamma), \gamma \sim \pi(\gamma) \). In this case, we construct the Gibbs sampler

\[
\beta_0 \sim \pi(\beta_0 | y, \gamma), \\
\gamma \sim \pi(\gamma | y, \beta_0).
\]