The Gibbs sampler was developed by Geman & Geman (1984). Let \( Z_i = (X_i, Y_i)' \) be a Markov chain. The Gibbs sampler can be used to generate specific multivariate distributions. Let \( f(x, y) \) be a given joint density and \( f(x|y) \) and \( f(y|x) \) to be conditional densities. The Gibbs sampling algorithm is given by

1. Generate \( Z_0 = (X_0, Y_0)' \). Set \( i = 1 \).
2. Generate \( X_i \sim f(x_i|Y_{i-1} = y_{i-1}) \)
3. Generate \( Y_i \sim f(y_i|X_i = x_i) \)
4. Set \( i = i + 1 \) and goto step 2.

2. **Theorem.** \( X_i, Y_i, (X_i, Y_i) \) generated by the Gibbs sampler are Markov chains.

   **Proof.** For chain \( X_i \), the transition kernel is

   \[
   K_X(x, x^*; y) = \int f(y|x) f(x^*|y) \, dy.
   \]

   Since \( X_i \) is simulated conditionally only on \( x_{i-1} \), \( \{X_i\} \) is a Markov chain. The invariant distribution of \( X_i \) is marginal distribution \( f_X(x) = \int f(x, y) \, dy \).

3. Let us simulate bivariate normal

   \[
   Z = (X, Y)' \sim N(0, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix})
   \]

   The joint density for \( Z \sim N(\mu, V) \) is given by

   \[
   f(z) = \frac{1}{2\pi \det V^{1/2}} \exp \left[ -\frac{zV^{-1}z}{2} \right].
   \]

   To simulate bivariate normal using the Gibbs sampler, we need to know the conditional distribution (we already know how to simulate multivariate normal based on the Cholesky factorization).

Then it can be shown that

\[
\frac{f(x|y)}{f(y)} = \frac{f(z)}{\int f(z) \, dx}.
\]

It can be shown that

\[
X|Y = y \sim N(\rho y, 1-\rho^2) \sim \rho y + \sqrt{1-\rho^2} N(0, 1),
\]

\[
Y|X = x \sim N(\rho x, 1-\rho^2) \sim \rho x + \sqrt{1-\rho^2} N(0, 1).
\]

\[
x(1)=0;y(1)=0;
rho=0.9;
c=sqrt(1-rho^2);
for i=2:6000
x(i)=rho*y(i-1)+ c*snrnd(1);
y(i)=rho*x(i)+ c*snrnd(1);
end;
subplot(1,2,1);
plot(x(1:50),y(1:50));
subplot(1,2,2);
plot(x(4000:6000),y(4000:6000),'.');