Stat 312: Lecture 04
Moment Matching

Moo K. Chung
mchung@stat.wisc.edu

September 14, 2004

1. Given a random sample $X_1, \cdots, X_n$, a linear estimator of parameter θ is an estimator of form
$$\hat{\theta} = \sum_{i=1}^{n} c_i X_i.$$
Then it can be shown that $\bar{X}$ is the MVUE for population mean among all linear unbiased estimators.
Proof. Case $n = 2$ will be proved. The general statement follows inductively. Consider linear estimators
$$\hat{\mu} = c_1 X_1 + c_2 X_2.$$
To be unbiased, $c_1 + c_2 = 1$. To be most efficient among all unbiased linear estimators, the variance has to be minimized. The variance is
$$\mathbb{V}\hat{\mu} = c_1^2 \mathbb{V}X_1 + c_2^2 \mathbb{V}X_2 = \left[c_1^2 + (1-c_1)^2\right]\sigma^2$$
The quadratic term in the bracket $2c_1^2 - 2c_1 + 1$ is minimized when $c_1 = 1/2$.

2. Given a random sample $X_1, \cdots, X_n$, the k-*th sample moment* is $M_k = \sum_{j=1}^{n} X_j^k / n$. The *moment estimators* of population parameters are obtained by matching the sample moments to correspond population moments and solving the resulting equations simultaneously.

3. *Exponential distribution.* X is an exponential distribution with parameter λ if the density function is $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. It can be shown that $\mathbb{X} = \frac{1}{\lambda}, \mathbb{V}X = \frac{1}{\lambda^2}$.

4. Given random sample $X_1, \cdots, X_n$, the *likelihood function* is given as the product of probability or density functions, i.e.
$$L(\theta) = f(x_1;\theta) f(x_2;\theta) \cdots f(x_n;\theta).$$
The *maximum likelihood estimatate* of θ is an estimate that maximizes $L(\theta)$. If we denote $\hat{\theta} = \theta(x_1, \cdots, x_n)$ to be the maximum likelihood estimate, The *maximum likelihood estimator* (MLE) of θ is denoted by $\hat{\theta} = \hat{\theta}(X_1, \cdots, X_n)$.

Review Problems.
Example 6.12. Example 6.16. Example 6.17.