Suppose we measure temperature $Y$ at position $x$ and time $t$ in a classroom $M \in \mathbb{R}^3$. Since every measurement will be error-prone, we model the temperature as

$$Y(x, t) = \mu(x, t) + \epsilon(x, t)$$

where $\mu$ is the unknown signal and $\epsilon$ is the measurement error. The measurement error can be modelled as a random variable. So at each point $(x, t) \in M \otimes \mathbb{R}^+$, measurement error $\epsilon(x, t)$ is a random variable. The collection of random variables

$$\{\epsilon(x, t) : (x, t) \in M \otimes \mathbb{R}^+\}$$

is called a stochastic process. A stochastic process that is indexed by a spatial variable is called a random field. A formal measure theoretic definition can be found in Geometry of Random Fields by Adler (1980) and Introduction to the Theory of Random Processes by Gikhman and Skorokhod (1969).

\section{Gaussian random fields}

A random vector $X = (X_1, \cdots, X_m)$ is multivariate normal if $\sum_i c_i X_i$ is Gaussian for every possible choice of $c_i$. A random fields $\epsilon(x) \in \mathbb{R}^N$ is a Gaussian random field if $\epsilon_1(x_1), \cdots, \epsilon_m(x_m)$ is multivariate normal for any $x_i \in M \subset \mathbb{R}^N$. $\epsilon$ is a mean zero Gaussian field if $\mathbb{E}\epsilon(x) = 0$ for all $x$. The covariance function of mean zero field is defined as

$$R(x, y) = \mathbb{E}\epsilon(x)\epsilon(y).$$

The variance field of $\epsilon$ is given by $R(x, x)$. Mean zero Gaussian field is completely characterized by the covariance function. A Gaussian random vector field is defined similarly.

$$e(x) = (e_1(x), \cdots, e_n(x))^t$$
is a $n$-dimensional Gaussian vector field if $e_i$ are Gaussian fields. We may generalize it further to matrix and tensor fields.

Two fields $e_1$ and $e_2$ are independent if $e_1(x)$ and $e_2(y)$ are independent for every $x$ and $y$. For mean zero Gaussian fields, $e_1$ and $e_2$ are independent if and only if the cross-covariance function

$$R(x, y) = \mathbb{E} e_1(x) e_2(y)$$

vanishes for all $x$ and $y$.

Let $\mathcal{G}$ be a collection of Gaussian random fields. Suppose $e_1, e_2 \in \mathcal{G}$. forms a vector space if $c_1 e_1 + c_2 e_2 \in \mathcal{G}$ for all $c_1$ and $c_2$. It is trivial to see that $\mathcal{G}$ is an infinite-dimensional vector space. However, there exists a finite vector space $\mathcal{G}_p$ that is the closest to $\mathcal{G}$ in the least-squares sense. For any linear operator $f$, $f(\mathcal{G}) \subset \mathcal{L}$. We show this for the derivatives of Gaussian fields.

## 2 Derivative Fields

The sequence of random fields $X_h(t)$ converges to $X(t)$ as $h \to 0$ in mean-square if

$$\lim_{h \to 0} \mathbb{E} |X_h(t) - X(t)|^2 = 0.$$ 

We denote it by $\lim_{h \to 0} X_h(t) = X(t)$ if there is no ambiguity. The convergence in the mean square implies the convergence in mean. Note that

$$\mathbb{E} |X_h(t) - X(t)|^2 = -\text{Var}[X_h(t) - X(t)]^2 - [\mathbb{E} X_h(t) - X(t)]^2.$$ 

Now let $X_h \to X$ in mean square. The left hand side is positive and converges to zero while the right hand side negative. Hence each term in the right hand side should converges to zero.

We define derivative in mean square as

$$\frac{dX(t)}{dt} = \lim_{h \to 0} \frac{X(t+h) - X(t)}{h}.$$ 

Note that if $X(t)$ and $X(t+h)$ are Gaussian random fields, $X(t+h) - X(t)$ is again Gaussian and hence the limit on the right hand side is again Gaussian. If $R$ is the covariance function of the mean zero Gaussian field $X$, the covariance function of its derivative is given by

$$\mathbb{E} \left[ \frac{dX(t)}{dt} \frac{dX(s)}{ds} \right] = \frac{\partial^2 R(t, s)}{\partial t \partial s}.$$ 

Partial derivatives are defined similarly.
3 Integration of Fields

We define the integration of a random field as the limit of Riemann sum of the field.

Let \( \bigcup_{i=1}^{n} \Omega_i \) be a partition of \( \Omega \subset \mathbb{R}^n \), i.e. \( \Omega = \bigcup_{i=1}^{n} \Omega_i \) and \( \Omega_i \cap \Omega_j = \emptyset \) if \( i \neq j \).

Let \( t_i \in \Omega_i \) and \( \mu(\Omega_i) \) be the volume of \( \Omega_i \). Then we define the integration of a random field as

\[
\int_{\Omega} X(t) \, dt = \lim_{\mu(\Omega_j) \to 0} \sum_{i=1}^{n} X(t_i) \mu(\Omega_i).
\]

Multiple integration is defined similarly. When we integrate a Gaussian field, it is the limit of a linear combination of Gaussian random variables so it is again Gaussian.

Consider the following integral

\[
Y(t) = \int K(t, s)X(s) \, ds.
\]

where \( K \) is called the kernel of the integral. Define convolution between kernel \( K \) and random field \( X \) as the above integral.

\[
Y(t) = K \ast X(t).
\]

Suppose the kernel to be isotropic probability density, i.e. \( K(t, s) = K(t - s) \) and \( \int K(t) \, dt = 1 \). Further we may assume \( K \) to be unimodal with some parameter \( \sigma \) such that

\[
\lim_{\sigma \to 0} K(t; \sigma) \to \delta(t),
\]

the Dirac-delta function. The Dirac-delta function is defined as a function that satisfies

\[
\delta(t) = 0, t \neq 0 \quad \text{and} \quad \int \delta(t) \, dt = 1.
\]

Since the Dirac-delta function satisfies

\[
\int \delta(t - s)f(s) \, ds = f(t),
\]

it can be easily seen that

\[
\lim_{\sigma \to 0} K(\cdot; \sigma) \ast X \to X.
\]

Theory on linear filter will continue.