Medical Image Analysis

Instructor: Moo K. Chung
mchung@stat.wisc.edu

Lecture 17.
Fractal Dimension Analysis

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Image Complexity

• Motivation: why we are interested in studying image complexity?

• Image complexity can characterize an image and the underlying clinical statue (existence of cancer, Alzheimer's disease).

• The most widely used image complexity measure is the fractal dimension.

• We will study fractals and fractal dimension.
Use of Fractals

• Quantification of biological systems (most complex and chaotic in science).

• Fractal image compression: it can achieve the compression ratio of up to 600:1.

• Fractal rendering: realistic computer rendering of clouds, rocks, shadows
Examples of fractals

Fractals can be generated by recursive formula in a complex plane.

For instance, the following formula will generate the left fractal:

\[ Z_{n+1} = Z_n + C \]
History of fractals

• Benoit B. Mandelbrot (1960). *Fractal geometry of nature*

• Mandelbrot termed *fractals* for geometric objects with self similarity.
Properties of Fractals

• Infinite detail at every point of the object

• Self (affine) similarity between parts and overall features of the object.

• Zoom into traditional shape: less detail

• Zoom into fractals: more detail
Deterministic nature of fractals

- Fractals are based on recursive mathematical formula: it can be predicted.

- Fractals are made to predict a complex and chaotic system deterministically.
Cardiovascular system

Similar to Pythagor tree
Euclidean Dimension (ED)
number of independent parameters that describes an object

• Dimension of an object measures the complexity of the object. 3D object is more complex than 1D object in general.

• Euclidean dimension: nonnegative integers 0, 1, 2, 3, …

• Fractal dimension: 1.56, 2.49, 3.45, …
Fractal Dimension (FD)

- amount of variations in the object

- Measure of roughness/fragmentation of the object

Small FD = less jagged/complex
Large FD = more jagged/complex
Constructing Fractals

- Deterministic fractals are constructed by an iterative process with the initial configuration.

**Sierpinski lattice** is constructed from a large triangle by recursively cutting smaller lattice.

$$FD = \frac{\ln 3}{\ln 2} = 1.58496...$$
Sierpinski Triangle

Sierpinski Sponge

FD = \frac{\ln 9}{\ln 3} = 2
Koch curve

FD = log_3(4) = 1.261859...

Hilbert curve

FD = 2
Example of self-similarity

Dilate by a factor of $k=2$
$N=4$ copies of the self similar original square.

Dilate by a factor of $k=3$
$N=9$ copies of the self similar original square.

$K=\text{scale factor}$
$N=\text{number of copies}$. 
Measure of self-similarity

- For the square, we have

\[ k^2 = N \]

- Alternately,

\[ \log_k N = 2 \]

Euclidean dimension
1D example

Line segment

Original

Dilated

$k = \text{scale factor} = 2$

$N = \text{number of copies} = 2$

$$\log_k N = 1$$

Euclidean dimension
3D example

Cube

Original

Dilated

$k = \text{scale factor} = 2$

$N = \text{number of copies of original} = 8$

$\log_k N = 3$

Euclidean dimension
ED to FD

- $\log_k N$ measures the dimension of the object.

- This is the definition of the *dimension of a self-similar object*.

- We call this dimension as the *fractal dimension*. 
FD computation for Sierpinski carpet

\[ \log_3 8 \approx 1.89 \]
Estimating FD

- FD can be explicitly computed on fractals that are mathematical given.

- In practice, the object of interest may not be a fractal but we can still estimate FD.

- Box-counting dimension (Mandelbrot). The most widely used method

- Perimeter-area dimension (Hausdorff)
Dimension of data

• FD can be taken as the approximation to the intrinsic dimension ($D_i$) of data, which can be different from the embedding dimension ($D_e$).

\[
\text{count} \left( x, x' : \| x - x' \| < k \right) \approx k^{FD}
\]

- line in a plane:
  - $D_e = 2$
  - $D_i = 1$
  - $FD \approx 1$

- uniform dist. in a plane:
  - $D_e = 2$
  - $D_i = 2$
  - $FD \approx 2$
FD and PCA

FD and the number of significant principal component?
FD and Factor Analysis

FD and the number of significant factors?
FD and evolution

• Tree branching is related to evolution. There is a direct relationship between branching patterns and common ancestry (Bickel, 2000)
## FD Analysis Results

<table>
<thead>
<tr>
<th>Type</th>
<th>Domestic</th>
<th>Wild</th>
</tr>
</thead>
<tbody>
<tr>
<td>Birch</td>
<td>1.262</td>
<td>1.113</td>
</tr>
<tr>
<td>Cherry</td>
<td>1.647</td>
<td>1.459</td>
</tr>
<tr>
<td>Oak</td>
<td>1.648</td>
<td>1.279</td>
</tr>
<tr>
<td>Maple</td>
<td>1.378</td>
<td>1.358</td>
</tr>
<tr>
<td>Dogwood</td>
<td>1.656</td>
<td>1.456</td>
</tr>
<tr>
<td>Poplar</td>
<td>1.424</td>
<td>1.456</td>
</tr>
</tbody>
</table>
FD computation by box counting

Gray scale image

Binary image: pixels above a certain threshold are set to one

Boundary of the object is obtained
Box counting process

Break the image into boxes of a given size and count how many of those boxes contain the contour.

We perform this process for several different box sizes.

If a box contains the contour, it is colored white.
FD estimation

<table>
<thead>
<tr>
<th>Box size (in pixels)</th>
<th>number of boxes containing the contour</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6544</td>
</tr>
<tr>
<td>3</td>
<td>3897</td>
</tr>
<tr>
<td>6</td>
<td>1562</td>
</tr>
<tr>
<td>12</td>
<td>591</td>
</tr>
<tr>
<td>22</td>
<td>250</td>
</tr>
<tr>
<td>40</td>
<td>101</td>
</tr>
<tr>
<td>75</td>
<td>32</td>
</tr>
<tr>
<td>138</td>
<td>9</td>
</tr>
<tr>
<td>256</td>
<td>4</td>
</tr>
</tbody>
</table>

FD = slope = 1.5439
Schedule for remaining classes

April 26 (Thursday). no class. Emotion symposium.
May 1 (Tuesday). Fractal analysis
May 3 (Thursday). Overview
May 8 (Tuesday). Student presentation.
May 10 (Thursday). Student presentation.
**May 16 12:15pm.** The deadline for research reports/literature review reports. After the deadline, reports will not be accepted (After the grade submission, I have to leave to LA immediately).
Schedule for Student presentation
Requirement for 3 credit students
15min presentation + 5 min discussions
Presentation will account for 20% of your final grade

• **Jamie L. Hanson** (VBM and effect of template)
• **Elizabeth B. Huchinson** (??????)
• **Daniel L.B. Levinson** (VBM in what population ?)
• **David M. Perlman** (fMRI analysis on cortex ?)
• **Brianna S. Schuyler** (cortical thickness analysis)
• **Ammet B. Soni** (fMRI and machine learning)
• **Shubing Wang** (Fourier transform)
• **Daniel J. Kelley** (Brain asymmetry analysis)
Schedule for presentation

Please send PDF or PowerPoint to me via email in the morning of presentation to save time for switching laptops. If you go to PowerPoint print option, you can generate PDF (Apple laptop will be used and there are known conflict between Apple PowerPoint and Windows PowerPoint)

• May 8
  11:00-11:20
  11:20-11:40
  11:40-12:00
  12:00-12:20

• May 10
  11:00-11:20
  11:20-11:40
  11:40-12:00
  12:00-12:20