Math 340–Problem Solving Seminar, Fall 2001, Problem Set 2

(1) Starting with any three-digit number \( n \) (such as \( n = 625 \)), we obtain a new number \( f(n) \) which is equal to the sum of the three digits of \( n \), their three products in pairs, and the product of all three digits.

(a) Find the value of \( \frac{n}{f(n)} \) when \( n = 625 \). (The answer is an integer!)

(b) Find all three-digit numbers \( n \) such that the ratio \( \frac{n}{f(n)} = 1 \).

(2) The sequence of integers \( u_0, u_1, u_2, u_3, \ldots \) satisfies \( u_0 = 1 \) and

\[
u_{n+1}u_{n-1} = ku_n \quad \text{for each} \quad n \geq 1,
\]

where \( k \) is some fixed positive integer. If \( u_{2000} = 2000 \), determine all possible values of \( k \).

(3) Let \( a \) and \( b \) be positive integers. Prove that

\[
4(a^3 + b^3) \geq (a + b)^3
\]

(4) Given any real number \( a \neq -1 \), the sequence \( x_1, x_2, x_3, \ldots \) is defined by

\[x_1 = a \quad \text{and} \quad x_{n+1} = x_n^2 + x_n \quad \text{for all} \quad n \geq 1.\]

Let

\[y_n = \frac{1}{1 + x_n}, \quad S_n = \sum_{i=1}^{n} y_i, \quad P_n = \prod_{i=1}^{n} y_i.
\]

Prove that

\[aS_n + P_n = 1 \quad \text{for all} \quad n \geq 1.
\]

(HINT: First show that \( P_n = \frac{a}{x_{n+1}} \).)

(5) Given that \( x \) is a positive integer, find all solutions of

\[
\left\lfloor \sqrt{1} \right\rfloor + \left\lfloor \sqrt{2} \right\rfloor + \cdots + \left\lfloor \sqrt{x^3 - 1} \right\rfloor = 400.
\]

Note: \([z]\) denotes the largest integer \( \leq z \).