Introduction to Statistics for the Life Sciences

Spring 2015
Chapter 3: Random Variables. (pages 102 - 115)
Note: I deviate from the book substantially here.
For now, skip 3.4. This lecture is based on 3.5 and 3.6.
The expectation or mean of a discrete random variable $Y$ is given by

$$
\mu_Y = E(Y) = \sum_y yP(Y = y)
$$

where $y$ ranges over the values that $Y$ takes.
**Definition**

**Variance of a Random Variable**

The variance of a random variable $Y$ is denoted $\sigma_Y^2$ or $V(Y)$ and is given by

$$
\sigma_Y^2 = V(Y) = E[(Y - \mu_Y)^2] = \sum_y (y - \mu_Y)^2 P(Y = y)
$$

where $y$ ranges over the values that $Y$ takes.

- Can $\sigma_Y^2$ be negative?
- If clear from context, we will write $\sigma^2$ instead of $\sigma_Y^2$. 

STAT 371
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Standard Deviation of a Random Variable

The standard deviation of a random variable $Y$ is denoted $\sigma_Y$ or $SD(Y)$ and is given by

$$\sigma_Y = \sqrt{\sigma_Y^2}$$

where $y$ ranges over the values that $Y$ takes.
Example

Suppose $Y$ is a random variable taking the value 0 with $P(Y = 0) = 1/2$ and 1 with $P(Y = 1) = 1/2$. Find $\sigma^2$.

Since $\sigma^2 = \sum_y (y - \mu)^2 P(Y = y)$, we need to find $\mu$ first.

$\mu = \sum_y y P(Y = y) = (0) P(Y = 0) + (1) P(Y = 1) = 0 + (1)(1/2) = 1/2$.

Then

$$\sigma^2 = (1 - 1/2)^2 P(Y = 0) + (0 - 1/2)^2 P(Y = 1) = (1/2)^2 P(Y = 0) + (-1/2)^2 P(Y = 1) = (1/4) P(Y = 0) + (1/4) P(Y = 1) = (1/4)(1/2) + (1/4)(1/2) = 1/4$$
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In the previous example, \( \sigma^2 = 1/4 \). So \( \sigma = \sqrt{1/4} = 1/\sqrt{4} = 1/2 \).
Shortcut for Variance of a Random Variable

For a random variable $Y$, 

$$\sigma^2 = \left[ \sum_y y^2 P(Y = y) \right] - \mu^2$$
Example

Suppose $Y$ takes the value 1 with $P(Y = 1) = 1/3$ and the value $-1$ with $P(Y = -1) = 2/3$. Find $\sigma^2$ with the shortcut

$$
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- $\mu = \sum_y y P(Y = y) = (-1)P(Y = -1) + (1)P(Y = 1) = (-1)(2/3) + (1)(1/3) = -1/3.$
- $\sum_y y^2 P(Y = y) = (-1)^2 P(Y = -1) + (1)^2 P(Y = 1) = 1/3 + 2/3 = 1.$
- Then $\sigma^2 = 1 - (-1/3)^2 = 1 - 1/9 = 8/9$.
- $\sigma = \sqrt{8/9} = \frac{2\sqrt{2}}{3}$.
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- $\sigma = \sqrt{8/9} = \frac{2\sqrt{2}}{3}$
Interpretation of $\sigma^2$

- Textbook defines $\mu$ and $\sigma^2$ but doesn’t discuss their meaning.
- $\mu$: we already know this is the “true” mean.
- $\sigma$ is the “true” standard deviation; so $\sigma$ is the “true” typical distance of the values of $Y$ from $\mu$. 
A trial is an experiment with two outcomes, success and failure, where the probability of success is some number $p$ such that $0 < p < 1$. We consider a sequence of $n$ such trials, which are identical in the sense that the probability of success $p$ is the same for each trial.

Example:
- Call heads a “success” and tails a “failure”; then consider $n = 6$ tosses of a fair coin.
- $p = 1/2$
- This experiment could be called a sequence of 6 trials.
Definition

**Sequence of Trials**

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If a sequence of $n$ trials has the property that the outcome of any given trial has no effect on the outcome of subsequent trials, then we say the trials are independent. We sometimes call such a sequence of $n$ independent trials Bernoulli trials.

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Example: Call heads a “success” and tails a “failure”; then consider $n = 6$ tosses of a fair coin. The trials are independent.
Suppose you’re on the street in NYC. A street magician tosses a coin which lands on heads 10 times in a row. S/he asks you to bet on the outcome of the next flip. What do you do?

- You do not trust that the next coin toss is fair and walk away.
- Suppose everything was fair and right.
- How do you bet?
- Because the coin tosses are independent, $P(\text{heads}) = 1/2$ despite the past outcomes.
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A binomial random variable is a random variable that satisfies the following conditions:

- **Binary outcomes**: There are two possible outcomes for each trial (success and failure).
- **Independent trials**: The outcomes of the trials are independent of each other.
- **$n$ is fixed**: The number of trials is set ahead of time.
- **Same value of $p$**: The probability of success is the same for all trials.

Examples of this?
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Examples of this?
Definition

Binomial Distribution Formula

For a binomial random variable $Y$, the probability that $n$ trials result in $j$ successes is given by

$$P(Y = j) = \binom{n}{j} p^j (1 - p)^{n-j}$$

for $j = 0, 1, 2, ..., n$. 
Combinations

If $n$ and $j$ are numbers such that $j \leq n$, then we say “$n$ choose $j$” and write

$$\binom{n}{j} = \frac{n!}{j!(n-j)!},$$

where $j! = j \times (j - 1) \times \ldots \times 1$. Define $0! = 1$.

Interpretation of combinations: number of ways $n$ objects can be put into a group of size $j$. 
Combination Examples

\[
\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4!}{2!2!} = \frac{4 \times 3 \times 2 \times 1}{(2 \times 1)(2 \times 1)} = \frac{4 \times 3}{2} = 2 \times 3 = 6
\]
\[ \begin{align*} \binom{4}{3} &= \frac{4!}{3!(4 - 3)!} \\ &= \frac{4!}{3!1!} \\ &= \frac{4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)(1)} \\ &= 4 \end{align*} \]
Combination Examples

\[
\binom{4}{1} = \frac{4!}{1!(4 - 1)!}
\]

\[
= \frac{4!}{1!3!}
\]

\[
= \frac{4!}{3!1!}
\]

\[
= \binom{4}{3}
\]

\[
= 4
\]
Combination Examples

Remember $0! = 1$.

\[
\binom{4}{4} = \frac{4!}{4!(4 - 4)!} = \frac{4!}{4!0!} = \frac{1}{0!} = 1
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\binom{4}{0} = \frac{4!}{0!(4 - 0)!} = \frac{4!}{0!4!} = \binom{4}{4} = 1
\]
The textbook uses the *nonstandard* notation of $nC_j$. Note that

$$nC_j = \binom{n}{j}$$
Let $Y$ be a binomial random variable with $n = 4$, where $p$ is an unknown fixed number. Construct a table that expresses $P(Y = y)$ in terms of $p$ for $y = 0, 1, 2, 3, 4$.

Solution: We know that $P(Y = y) = \binom{4}{y} p^y (1 - p)^{4-y}$ for $y = 0, 1, 2, 3, 4$. Therefore...
### Binomial Example

<table>
<thead>
<tr>
<th>No. successes</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\binom{4}{0} p^0 (1 - p)^{4-0} = (1 - p)^4$</td>
</tr>
<tr>
<td>1</td>
<td>$\binom{4}{1} p^1 (1 - p)^{4-1} = 4p(1 - p)^3$</td>
</tr>
<tr>
<td>2</td>
<td>$\binom{4}{2} p^2 (1 - p)^{4-2} = 6p^2(1 - p)^2$</td>
</tr>
<tr>
<td>3</td>
<td>$\binom{4}{3} p^3 (1 - p)^{4-3} = 4p^3(1 - p)$</td>
</tr>
<tr>
<td>4</td>
<td>$\binom{4}{4} p^4 (1 - p)^{4-4} = p^4$</td>
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Consider the case of the number of heads in 6 coin tosses. Calculate $\mu$.

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>0.00</td>
<td>0.02</td>
</tr>
<tr>
<td>1.00</td>
<td>0.09</td>
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<tr>
<td>2.00</td>
<td>0.23</td>
</tr>
<tr>
<td>3.00</td>
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</tr>
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The sum of the last column is $\mu = 3$. Also, $np = (6)(1/2) = 3$. 
Result

**Expectation of a Binomial Random Variable**

Let $Y$ be a binomial random variable with $n$ trials and probability of success $p$. Then the expected number of successes is

$$
\mu_Y = np,
$$

or the number of trials multiplied by the success probability.

In other words, if we know $Y$ is binomial, we don't have to use

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\mu_Y = \sum_{y} y P(Y = y) = \sum_{y=0}^{n} y \binom{n}{y} p^y (1 - p)^{n-y}
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Let $Y$ be a random variable taking the value 1 with $P(Y = 1) = 1/2$ and 0 with $P(Y = 0) = 1/2$. We already showed $\mu = 1/2$. Do so with the binomial distribution.

- $Y$ is equivalent to a binomial random variable with $n = 1$ and $p = 1/2$.
- Therefore $\mu = np = (1)(1/2) = 1/2$.
- Much easier than using $\mu = \sum_y yP(Y = y)$!
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