Discussion Session # 5

Homework #3:
Page 102 (ex4a): 1, 5
Page 109 (ex4b): 2, 4
Page 113 (ex4c): 1, 2, 3
Page 117 (misc4): 1, 2, 4
Page 136 (misc5): 4, 6
Page 148 (ex6a): 1
Page 185 (misc7): 1, 2

There’s a typo in exercise 4b, # 2, (a): The correlation coefficient of $\hat{\beta}_0$ and $\hat{\beta}_1$ is $-\bar{x}/\sqrt{n\sum x_i^2}$.

1 The F-test and hypothesis testing

1.1 Exercise 4b, # 5
Given $Y = \theta + \varepsilon$, where $\varepsilon \sim N(0, \sigma^2 I_4)$, and $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 0$, show that the F-statistic for testing $H_0: \theta_1 = \theta_3$ is
\[
\frac{2(Y_1 - Y_3)^2}{(Y_1 + Y_2 + Y_3 + Y_4)^2}
\]

1.2 Miscellaneous Exercise 4, # 2
Given two regression lines
\[
Y_{ki} = \beta_k x_i + \varepsilon_{ki}, \quad k = 1, 2; i = 1, \ldots, n
\]
show that the F-statistic for testing $H_0: \beta_1 = \beta_2$ can be put in the form
\[
F = \frac{(\hat{\beta}_1 - \hat{\beta}_2)^2}{2S_2^2(\sum_i x_i^2)^{-1}}
\]

2 Multiple Correlation Coefficient

2.1 Exercise 4c, # 1
Suppose that $\beta_1 = \beta_2 = \ldots = \beta_{p-1} = 0$. Find the distribution of $R^2$ and hence prove that
\[
\mathbb{E}(R^2) = \frac{p-1}{n-1}
\]

3 Sample Midterm Question: Question # 2 from 2011 midterm
Let $Y \sim N(X\beta, \sigma^2 V)$, where $X$ is a $n \times p$ matrix of rank $p$ and $V$ is a known positive-definite $n \times n$ matrix.
If $\beta^*$ is the generalized least squares estimate of $\beta$, prove that
(a) $Q = (Y - X\beta^*)'V^{-1}(Y - X\beta^*)/\sigma^2 \sim \chi^2_{n-p}$
(b) $Q$ is the quadratic nonnegative unbiased estimate of $(n - p)\sigma^2$ with minimum variance.
(c) If $Y^* = X\beta^* = P^* Y$, then $P^*$ is idempotent but not, in general, symmetric. For $n = 2, p = 2$, give a numeric example of $X$ and $V$ such that $P^*$ is asymmetric.