Discussion Session # 2

1 Homework

Assignment #1:

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Due in class on Sept. 19, 2012

2 Expectation and Variance of Quadratic Forms

The expectation of a vector of random variables $X$ is

$$E(X) = E\left[\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}\right] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{bmatrix}$$

If $A$, $B$, $C$ are conformable matrices, then

$$E(AXB + C) = AE(X)B + C$$

The variance-covariance matrix of $X$ is

$$\text{Var}[X] = \begin{bmatrix} E[(X_1 - \alpha_1)^2] & E[(X_1 - \alpha_1)(X_2 - \alpha_2)] & \cdots & E[(X_1 - \alpha_1)(X_n - \alpha_n)] \\
E[(X_2 - \alpha_2)(X_1 - \alpha_1)] & E[(X_2 - \alpha_2)^2] & \cdots & E[(X_2 - \alpha_2)(X_n - \alpha_n)] \\
\vdots & \vdots & \ddots & \vdots \\
E[(X_n - \alpha_n)(X_1 - \alpha_1)] & E[(X_n - \alpha_n)(X_2 - \alpha_2)] & \cdots & E[(X_n - \alpha_n)^2] \end{bmatrix}$$

where $\alpha_i = E(X_i)$.

If $X$ is a random vector, $A$ a symmetric matrix, $E(X) = \mu$ and $\text{Var}(X) = \Sigma$, then

$$E(X'AX) = \text{tr}(A\Sigma) + \mu'A\mu$$

2.1 Exercise 1a, # 4

If $X_1, X_2, \ldots, X_n$ are random variables satisfying $X_{i+1} = \rho X_i, i = 1, 2, \ldots, n - 1$, where $\rho$ is a constant and $\text{Var}(X_1) = \sigma^2$, find $\text{Var}(X)$. 

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2.2 Exercise 1b, # 1

Suppose that \(X_1, X_2\) and \(X_3\) are random variables with common mean \(\mu\) and variance matrix

\[
\text{Var}(X) = \sigma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/4 \\ 0 & 1/4 & 1 \end{pmatrix}
\]

Find \(E(X_1^2 + 2X_1X_2 - 4X_2X_3 + X_3^2)\).

2.3 Miscellaneous Exercises, # 3

Let \(X_1, X_2, \ldots, X_n\) be random variables with a common mean \(\mu\). Suppose that \(\text{Cov}(X_i, X_j) = 0\) for all \(i\) and \(j\) such that \(j > i + 1\). If

\[
Q_1 = \sum_{i=1}^{n} (X_i - \bar{X})^2
\]

and

\[
Q_2 = (X_1 - X_2)^2 + (X_2 - X_3)^2 + \ldots + (X_{n-1} - X_n)^2 + (X_n - X_1)^2
\]

prove that

\[
E\left(\frac{3Q_1 - Q_2}{n(n-3)}\right) = \text{Var}(\bar{X})
\]

3 Multivariate Probability Distributions and Moment Generating Functions

If \(\Sigma\) is positive definite, and \(\mu\) is a \(n\)-vector, then

\[
f(y) = \frac{1}{(2\pi)^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(y - \mu)'\Sigma^{-1}(y - \mu)\right\}
\]

Its moment generating function is

\[
\mathcal{M}(t) = \exp\left(t'\mu + \frac{1}{2}t'\Sigma t\right)
\]

3.1 Exercise 2a, # 1

Show that

\[
f(y_1, y_2) = k^{-1} \exp\left\{-\frac{1}{2} (2y_1^2 + y_2^2 + 2y_1y_2 - 22y_1 - 14y_2 + 65)\right\}
\]

is the density of a bivariate normal random vector \(\mathbf{Y} = (Y_1, Y_2)'\).

- Find \(k\).
- Find \(E(\mathbf{Y})\) and \(\text{Var}(\mathbf{Y})\).
3.2 Miscellaneous Exercises 2, # 1
Suppose that $\varepsilon \sim N_3(0, \sigma^2 I_3)$ and that $Y_0 \sim N(0, \sigma_0^2)$ independently of the $\varepsilon_i$. Define
\[ Y_i = \rho Y_{i-1} + \varepsilon_i, \quad i = 1, 2, 3. \]

- Find the variance-covariance matrix of $Y = (Y_1, Y_2, Y_3)'$.
- What is the distribution of $Y$.

3.3 Miscellaneous Exercises 2, # 11 (simplified)
Let $a_0, a_1, a_2, a_3$ be independent $N(0, \sigma^2)$ random variables and define
\[ Y_i = a_i + \phi a_{i-1}, \quad i = 1, 2, 3. \]
Show that $Y = (Y_1, Y_2, Y_3)'$ has a multivariate normal distribution and find its variance-covariance matrix.

4 Distribution of Quadratic Forms
Theorem 2.7: Let $Y \sim N_n(0, I_n)$ and $A$ be a symmetric matrix. Then $Y'AY \sim \chi^2_r$ if and only if $A$ is idempotent of rank $r$.

Theorem 2.8’s corollary: Let $Y \sim N_n(0, \Sigma)$ where $\Sigma$ is positive definite and $A$ be a symmetric matrix. Then $Y'AY \sim \chi^2_r$ if and only if $A \Sigma$ is idempotent of rank $r$.

4.1 Exercise 2d, # 3
If $Y \sim N_2(0, I_2)$, find values of $a$ and $b$ such that
\[ a(Y_1 - Y_2)^2 + b(Y_1 + Y_2)^2 \sim \chi^2_2 \]