

Intrinsic Aging and Classes of Nonparametric Distributions

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Introduction

Consider a nonnegative absolutely continuous random variable Y . The random variable Y could be the time to default in a credit risk (e.g., Ammann, 2001), the life time of a reliability system (e.g., Barlow and Proschan, 1975), or the demand for an item in a supply chain (e.g., Porteus, 2002). Nonparametric (aging) properties of the distribution function of this random variable often play a crucial role in characterizing the optimal operational policies associated with the random variable.

In classical reliability theory, important aging notions for a nonnegative absolutely continuous random variable Y are NBU (new better than used), NWU (new worse than used), IFR (increasing failure rate), IFRA (increasing failure rate on average), and the corresponding decreasing versions, DFR and DFRA.

Motivation

Recently, the importance of a new notion of aging for pricing problems has been recognized (Lariviere and Porteus, 2001). Here Y is the random valuation of a customer for a product, so

$$\bar{F}(p) = P(Y > p)$$

is the probability that a random customer will buy the product at price p , and

$$p\bar{F}(p)$$

is the expected revenue for price p . The aging notion appropriate for this application is based on the length-biased failure rate of the random variable Y , defined by

$$l(t) = th(t),$$

where $h(t) = f(t)/\bar{F}(t)$ is the usual hazard rate of Y and $f(t)$ is the density of Y . There will be a unique revenue maximizing price if Y is ILFR, i.e., if it is increasing in length-biased failure rate ($l(t)$ is increasing in t), and if $\lim_{x \downarrow a} l(x) \leq 1$ and $\lim_{x \uparrow b} l(x) > 1$ where (a, b) is the support of Y . Note that the ILFR property is weaker than the IFR property. Among other things, we introduce the notion of scaled conditional life,

$$Y_{\text{sc}}(y) = \{Y | Y > y\}/y,$$

and relate it to $l(t)$ — if Y is ILFR, then $Y_{\text{sc}}(y)$ is decreasing in y in the hazard rate sense.

Various Notions of Hazard Rates and Residual Lifetimes

Consider a nonnegative random variable Y . Let

$$a_Y = \inf\{y : F_Y(y) > 0\};$$

$$b_Y = \sup\{y : F_Y(y) < 1\}.$$

Residual life:

$$Y_R(y) = \{Y - y | Y > y\}, \quad a_Y \leq y < b_Y.$$

Hazard rate function:

$$h_Y(y) = \lim_{\Delta \downarrow 0} \frac{1}{\Delta} P\{Y_R(y) \leq \Delta\}, \quad a_Y \leq y < b_Y.$$

Cumulative hazard function:

$$H_Y(y) = -\log \bar{F}_Y(y), \quad a_Y \leq y < b_Y.$$

Note

$$h_Y(y) = \frac{d}{dy} H_Y(y), \quad a_Y \leq y < b_Y.$$

Conditional shortfall - inactivity time:

$$Y_S(y) = \{y - Y | Y \leq y\}, \quad a_Y \leq y < b_Y.$$

Reverse hazard rate function:

$$r_Y(y) = \lim_{\Delta \downarrow 0} \frac{1}{\Delta} P\{Y_S(y) \leq \Delta\}, \quad a_Y \leq y < b_Y.$$

Scaled conditional life:

$$Y_{SC}(y) = \frac{1}{y} \{Y | Y > y\} = \frac{y + Y_R(y)}{y}, \quad a_Y \leq y < b_Y.$$

Length-biased hazard rate function:

$$l_Y(y) = \lim_{\Delta \downarrow 0} \frac{1}{\Delta} P\{Y_{SC}(y) \leq 1 + \Delta\} = yh_Y(y),$$
$$a_Y \leq y < b_Y.$$

Scaled residual life:

$$Y_{SR}(y) = \frac{\{Y - y | Y > y\}}{b_Y - y} = \frac{Y_R(y)}{b_Y - y}, \quad a_Y \leq y < b_Y.$$

Scaled hazard rate function:

$$s_Y(y) = \lim_{\Delta \downarrow 0} \frac{1}{\Delta} P\{Y_{SR}(y) \leq \Delta\} = (b_Y - y)h_Y(y),$$
$$a_Y \leq y < b_Y.$$

Scaled hazard rates arise in the inspection contexts; see paper.

Classes of Functions

Let $\eta : [a, b] \rightarrow \mathbb{R}_+$ be a positive increasing function.

Super-additive [Sub-additive]: SupA [SubA]

$$\eta(x) + \eta(y) \leq [\geq] \eta(x + y), \quad x, y \in (a, b).$$

Star-Shaped [AntiStar-Shaped]: SS [AntiSS]

$$\frac{\eta(y)}{y} \text{ is increasing [decreasing] in } x \in (a, b).$$

Convex [Concave]: CX [CV]

Convex in Log Scale [Concave in Log Scale]: CX(Log) [CV(Log)]

$$\eta(e^y) \text{ is convex [concave].}$$

Log Convex [Log Concave]: LogCX [LogCV]

$$\log \eta(y) \text{ is convex [concave].}$$

Log Convex in Log Scale [Log Concave in Log Scale]: LogCX(Log) [LogCV(Log)]

$$\log \eta(e^y) \text{ is convex [concave].}$$

Scaled Convex [Scaled Concave]: ScCX [ScCV]

$$(b - y)\eta'(y) \text{ is increasing [decreasing] in } x \in (a, b).$$

Since we only consider increasing functions, we insert an I and write, e.g.,

$$\text{ISupA.}$$

In some cases we also want to require $a = 0$ and $\eta(0) = 0$, in which case we write, e.g.,

$$\text{ISupA}_0.$$

When we restrict ourselves to functions $\eta : (a, b) \rightarrow \mathbb{R}_+$ where $b < \infty$, we write, e.g.,

$$\text{ICX}^b.$$

Sometimes we want functions such that $\lim_{x \rightarrow b} \eta(x) = \infty$, in which case we write, e.g.,

$$\text{ICX-}\infty.$$

Lemma 1.

$$\begin{array}{lll} \text{ICX}_0 & \implies & \text{ISS}_0 & \implies & \text{ISupA}_0 \\ \text{ICV}_0 & \implies & \text{IAntiSS}_0 & \implies & \text{ISubA}_0 \\ \text{ILogCX} & \implies & \text{ILogCX(Log)} & \implies & \text{ICX(Log)} \\ \text{ILogCX} & \implies & \text{ICX} & \implies & \text{ICX(Log)} \\ \text{ICV(Log)} & \implies & \text{ILogCV(Log)} & \implies & \text{ILogCV} \\ \text{ICV(Log)} & \implies & \text{ICV} & \implies & \text{ILogCV} \\ \text{IScCX}^b & \implies & \text{ICX}^b & & \\ \text{ICV}^b & \implies & \text{IScCV}^b & & \end{array}$$

Classes of Distributions Based On Aging

The random variable Y is said to be

New Better (Worse) than Used: NBU [NWU]

$$\bar{F}_Y(x)\bar{F}(y) \geq [\leq] \bar{F}(x+y), \quad x, y \in \mathbb{R}_+$$



$$H_Y \in \text{ISupA}_{0-\infty} [\text{ISubA}_{-\infty}]$$

Increasing (Decreasing) Failure Rate Average:

IFRA [DFRA]

$$H_Y \in \text{ISS}_{0-\infty} [\text{IAntiSS}_{-\infty}]$$

Increasing (Decreasing) Failure Rate: IFR [DFR]

$$H_Y \in \text{ICX}_{0-\infty} [\text{ICV}_{-\infty}]$$

Increasing (Decreasing) Reverse Failure Rate:

IRF [DRF]

$r_Y(y)$ is increasing [decreasing] in y

Increasing (Decreasing) Length-biased Failure Rate:

ILFR [DLFR]

$l_Y(y)$ is increasing [decreasing] in y

\Updownarrow

$H_Y \in \text{ICX}(\text{Log})-\infty$ [$\text{ICV}(\text{Log})_0-\infty$]

Increasing (Decreasing) Failure Rate Relative to Cumulative Hazard Rate:

IFR/C [DFR/C]

$h_Y(y)/H_Y(y)$ is increasing [decreasing] in y

\Updownarrow

$H_Y \in \text{ILogCX}-\infty$ [$\text{ILogCV}-\infty$]

Increasing (Decreasing) Failure Rate Relative to Average Hazard Rate:

IFR/A [DFR/A]

$H_Y \in \text{ILogCX}(\text{Log})-\infty$ [$\text{ILogCV}(\text{Log})-\infty$]

Increasing (Decreasing) Scaled Hazard Rate:

ISFR [DSFR]

$s_Y(y)$ is increasing [decreasing] in y

\Updownarrow

$H_Y \in \text{IScCX}^b$ [IScCV^b]

Relating Classes of Distributions

It is well-known and easy to show that

$$\begin{aligned} \text{IFR} &\implies \text{IFRA} \implies \text{NBU} \\ \text{DFR} &\implies \text{DFRA} \implies \text{NWU} \end{aligned}$$

Here we extend these relationships for the new classes of distributions.

Lemma 2.

$$\begin{aligned} \text{IFR/C} &\implies \text{IFR/A} \implies \text{ILFR} \\ \text{IFR/C} &\implies \text{IFR} \implies \text{ILFR} \\ \text{ISFR} &\implies \text{IFR} \\ \text{DLFR} &\implies \text{DFR/A} \implies \text{DFR/C} \\ \text{DLFR} &\implies \text{DFR} \implies \text{DFR/C} \\ &\text{DFR} \implies \text{DSFR} \end{aligned}$$

Some properties of $Y_{\mathbf{R}}$, $Y_{\mathbf{SC}}$, and $Y_{\mathbf{SR}}$

Here is a sample of results:

- (i) $Y \in \text{IFR} \iff Y_{\mathbf{R}}(y) \in \text{IFR} \quad \forall y \in [a_Y, b_Y]$.
- (ii) $Y \in \text{DFR} \iff Y_{\mathbf{R}}(y) \in \text{DFR} \quad \forall y \in [a_Y, b_Y]$.
- (iii) $Y \in \text{ILFR} \implies Y_{\mathbf{R}}(y) \in \text{ILFR} \quad \forall y \in [a_Y, b_Y]$.
- (iv) For $y \in [a_Y, b_Y]$, $Y \in \text{IFR} \iff Y_{\mathbf{SC}}(y) \in \text{IFR}$.
- (v) For $y \in [a_Y, b_Y]$, $Y \in \text{ILFR} \iff Y_{\mathbf{SC}}(y) \in \text{ILFR}$.
- (vi) For $y \in [a_Y, b_Y]$, $Y \in \text{IFR} \iff Y_{\mathbf{SR}}(y) \in \text{IFR}$.
- (vii) $Y \in \text{ILFR} \implies Y_{\mathbf{SR}}(y) \in \text{ILFR} \quad \forall y \in [a_Y, b_Y]$.

More results of this type, as well as results involving monotonicities with respect to various stochastic orders, can be found in the paper.

Intrinsic Life and Actual Life

Let X be a nonnegative absolutely continuous random variable. Suppose X is the intrinsic life of a reliability system. The actual life time T of this system will depend on how the intrinsic age is accumulated over the calendar time. For example, under extreme conditions the system will age faster than under milder conditions. Suppose the intrinsic age of the system at time t is $\phi(t)$ ($\phi(0) = 0$). Then

$$T = \inf\{t : \phi(t) \geq X; t \in \mathbb{R}_+\} =: \phi^{-1}(X),$$

and $X = \phi(T)$.

The following results are well known.

Proposition 1.

$$\begin{aligned} X \in \text{NBU} &\iff T \in \text{NBU} \forall \phi \in \text{ISupA} \\ X \in \text{NWU} &\iff T \in \text{NWU} \forall \phi \in \text{ISubA} \\ X \in \text{IFRA} &\iff T \in \text{IFRA} \forall \phi \in \text{ISS} \\ X \in \text{DFRA} &\iff T \in \text{DFRA} \forall \phi \in \text{IAntiSS} \\ X \in \text{IFR} &\iff T \in \text{IFR} \forall \phi \in \text{ICX} \\ X \in \text{DFR} &\iff T \in \text{DFR} \forall \phi \in \text{ICV} \end{aligned}$$

Here are new results.

Proposition 2.

$$\begin{aligned}
X \in \text{ILFR} &\iff T \in \text{ILFR} \forall \phi \in \text{ILogCX}(\text{Log}) \\
X \in \text{DLFR} &\iff T \in \text{DLFR} \forall \phi \in \text{ILogCV}(\text{Log}) \\
X \in \text{IFR}/\text{C} &\iff T \in \text{IFR}/\text{C} \forall \phi \in \text{ICX} \\
X \in \text{DFR}/\text{C} &\iff T \in \text{DFR}/\text{C} \forall \phi \in \text{ICV} \\
X \in \text{IFR}/\text{A} &\iff T \in \text{IFR}/\text{A} \forall \phi \in \text{ILogCX}(\text{Log}) \\
X \in \text{DFR}/\text{A} &\iff T \in \text{DFR}/\text{A} \forall \phi \in \text{ILogCV}(\text{Log}) \\
X \in \text{IFR}/\text{C} &\iff T \in \text{IFR}/\text{C} \forall \phi \in \text{ILogCX} \\
X \in \text{DFR}/\text{C} &\iff T \in \text{DFR}/\text{C} \forall \phi \in \text{ILogCV}
\end{aligned}$$

$$\begin{aligned}
X \in \text{IFR} &\implies T \in \text{ILFR} \forall \phi \in \text{ICX}(\text{Log}) \\
X \in \text{IFR}/\text{C} &\implies T \in \text{IFR}/\text{A} \forall \phi \in \text{ICX}(\text{Log}) \\
X \in \text{IFR} &\implies T \in \text{ISFR} \forall \phi \in \text{IScCX} \\
X \in \text{DFR} &\implies T \in \text{DSFR} \forall \phi \in \text{IScCV}-\infty
\end{aligned}$$

Various results in the literature are special cases of the results above. Here is another proposition that follows from more general results of ours. The first part below was proven in Lariviere (2006).

Proposition 3.

$$\begin{aligned}
X \in \text{ILFR} &\iff \log(X) \in \text{IFR} \\
X \in \text{IFR}/\text{A} &\iff \log(X) \in \text{IFR}/\text{C}
\end{aligned}$$

Generating Classes of Random Variables

Proposition 4. *Let $Z \sim \exp(1)$. Then*

$$X \in \text{NBU} \iff X = \psi(Z) \text{ for some } \psi \in \text{ISubA}$$

$$X \in \text{NWU} \iff X = \psi(Z) \text{ for some } \psi \in \text{ISupA}$$

$$X \in \text{IFRA} \iff X = \psi(Z) \text{ for some } \psi \in \text{IAntiSS}$$

$$X \in \text{DFRA} \iff X = \psi(Z) \text{ for some } \psi \in \text{ISS}$$

$$X \in \text{IFR} \iff X = \psi(Z) \text{ for some } \psi \in \text{ICV}$$

$$X \in \text{DFR} \iff X = \psi(Z) \text{ for some } \psi \in \text{ICX}$$

$$X \in \text{ILFR} \iff X = \psi(Z) \text{ for some } \psi \in \text{ILogCV}$$

$$X \in \text{DLFR} \iff X = \psi(Z) \text{ for some } \psi \in \text{ILogCX}$$

$$X \in \text{IFR/C} \iff X = \psi(Z) \text{ for some } \psi \in \text{ICV(Log)}$$

$$X \in \text{DFR/C} \iff X = \psi(Z) \text{ for some } \psi \in \text{ICX(Log)}$$

$$X \in \text{IFR/A} \iff X = \psi(Z) \text{ for some } \psi \in \text{ILogCV(Log)}$$

$$X \in \text{DFR/A} \iff X = \psi(Z) \text{ for some } \psi \in \text{ILogCX(Log)}$$

To characterize ILFR and DLFR random variables, it is more natural to use $Z \sim \text{Pareto}(1, 1) \in \text{ILFR} \cap \text{DLFR}$, which has

$$\begin{aligned}\bar{F}_Z(t) &= 1/t, & t \geq 1, \\ h_Z(t) &= 1/t, & t \geq 1, \\ l_Z(t) &= 1, & t \geq 1,\end{aligned}$$

Proposition 5. *Let $Z \sim \text{Pareto}(1, 1)$. Then*

$$\begin{aligned}X \in \text{ILFR} &\iff X = \psi(Z) \text{ for some } \psi \in \text{ILogCV}(\text{Log}) \\ X \in \text{DLFR} &\iff X = \psi(Z) \text{ for some } \psi \in \text{ILogCX}(\text{Log})\end{aligned}$$

Now consider IFR/C and DFR/C random variables. Let Z be a mixture of the constant 0 with probability $P\{Z = 0\} = 1 - 1/e$, and an absolutely continuous random variable with density

$$f(z) = e^z e^{-e^z}, \quad z > 0.$$

We have $Z \in \text{IFR/C} \cap \text{DFR/C}$.

Proposition 6. *Let Z be as above. Then*

$$\begin{aligned}X \in \text{IFR/C} &\iff X = \psi(Z) \text{ for some } \psi \in \text{ICV} \\ X \in \text{DFR/C} &\iff X = \psi(Z) \text{ for some } \psi \in \text{ICX}\end{aligned}$$