System Signatures in Dynamic Reliability Settings

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I. Introduction

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Since component lifetimes $X_1, \ldots, X_n$ of a system of order n are continuous random variables, the system's failure time will coincide with the value of a particular order statistic $X_{k:n}$.

II. System Signatures.

Defn. 2 (Samaniego (1985): Suppose a coherent system's n components have i.i.d. lifetimes $X_1, \ldots, X_n \sim F$, where F is a continuous distribution on $(0, \infty)$. The signature of the system, denoted by $\mathbf{s}$, is a vector in $[0, 1]^n$ whose ith element is $s_i = P(T = X_{i:n})$,where T is the failure time of the system.

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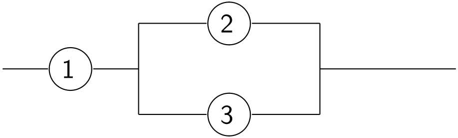
The i.i.d. assumption "levels the playing field" among systems and facilitates the comparison of the designs themselves. **The signature vector is a distribution-free measure of a system's design.**

II. System Signatures.

Example 1: A three component series-parallel system.



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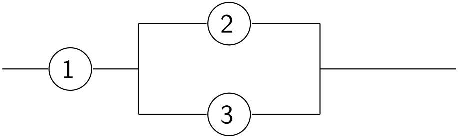

Ordered Component Failure Times	Order Statistic Equal to System Failure Time T
$X_1 < X_2 < X_3$	$X_{1:3}$
$X_1 < X_3 < X_2$	$X_{1:3}$
$X_2 < X_1 < X_3$	$X_{2:3}$
$X_2 < X_3 < X_1$	$X_{2:3}$
$X_3 < X_1 < X_2$	$X_{2:3}$
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It follows that this system has signature vector $\mathbf{s} = (1/3, 2/3, 0)$.

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$$\overline{F}_T(t) \equiv P(T > t) = \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} (F(t))^j (\overline{F}(t))^{n-j}.$$

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Note: Implicit in this result are the following useful relationships:

$$P(T > t) = \sum_{i=1}^{n} s_i P(X_{i:n} > t)$$

and

$$ET = \sum_{i=1}^{n} s_i EX_{i:n}.$$

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(i) If $\mathbf{s}_1 \leq_{\text{st}} \mathbf{s}_2$, then $T_1 \leq_{\text{st}} T_2$.

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(i) If $\mathbf{s}_1 \leq_{\text{st}} \mathbf{s}_2$, then $T_1 \leq_{\text{st}} T_2$.

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(i) If $\mathbf{s}_1 \leq_{\text{st}} \mathbf{s}_2$, then $T_1 \leq_{\text{st}} T_2$.

(ii) If $\mathbf{s}_1 \leq_{\text{hr}} \mathbf{s}_2$, then $T_1 \leq_{\text{hr}} T_2$.

(iii) If F is absolutely continuous and $\mathbf{s}_1 \leq_{\text{lr}} \mathbf{s}_2$, then $T_1 \leq_{\text{lr}} T_2$.

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$$\left\{ \mathbf{p} \in [0,1]^n : \sum_{i=1}^{n} p_i = 1 \right\}$$

is the signature of a mixed system, a fact that follows from the representation $\mathbf{p} = \sum_{k=1}^{n} p_k \mathbf{s}_{k:n}$.

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We note that the representation and preservation results above apply equally to coherent or mixed systems.

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Theorem 3: (Samaniego (2006)): Let $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ be the signature of a coherent or mixed system in n i.i.d. components with common lifetime distribution F. Then the coherent or mixed system with $(n+1)$-components with i.i.d. lifetimes $\sim F$ and corresponding to the signature vector $\mathbf{s}^*$ given by

$$\left(\tfrac{n}{n+1}s_1, \tfrac{1}{n+1}s_1 + \tfrac{n-1}{n+1}s_2, \tfrac{2}{n+1}s_2 + \tfrac{n-2}{n+1}s_3, \ldots, \tfrac{n-1}{n+1}s_{n-1} + \tfrac{1}{n+1}s_n, \tfrac{n}{n+1}s_n\right)$$

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For arbitrary $n < m$, the exact relationship between equivalent signatures of orders n and m is given in Navarro, Samaniego and Balakrishnan (2008).

III. Dynamic Signatures: Basic Notions

We treat the dynamic signature of a system, conditional on the information that exactly i components have failed by the inspection time t and that the system is working at time t. At inspection, the system may be viewed as a working used system whose failure will coincide with the failure of one of the remaining components. Let $E_i = \{X_{i:n} \leq t \leq X_{i+1:n}\}$, and assume that $P(E_i \cap \{T > t\}) > 0$. The dynamic signature of a system with signature **s**, given $E_i \cap \{T > t\}$, is simply the truncated signature given by

$$\mathbf{s}(n-i) = (s_{i+1}/S_i, s_{i+2}/S_i, \ldots, s_n/S_i)$$

where $S_i = \sum_{j=i+1}^{n} s_j$, and the argument $n - i$ is the order of the used system.

IV. Dynamic Versions of Aging

In reliability modeling, the usual notion of "New Better than Used" (NBU) utilizes a particular version of conditioning. Specifically, T is NBU if

$$P(T > x) \geq P(T > x + t \mid T > t) \quad \forall\, x > 0,\, \forall\, t > 0.$$

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We now extend this concept to that of a "uniformly new better than used" or UNBU system. Assume that the available information at time t is $\{T > t\}$ and $E_i = \{X_{i:n} \leq t < X_{i+1:n}\}$. The question we will now consider is when a system is conditionally NBU or uniformly NBU, where these concepts are defined as follows:

IV. Dynamic Versions of Aging

Definition 4: Consider a mixed system based on n components with i.i.d. lifetimes $X_1, \ldots, X_n \sim F$, where F is a continuous distribution on $(0, \infty)$. Let T be the system's lifetime, and let $E_i = \{X_{i:n} < t < X_{i+1:n}\}$, where $X_{0:n} \equiv 0$ and $X_{n+1:n} \equiv \infty$. For any fixed $i \in \{0, 1, 2, \ldots, n-1\}$, T is conditionally NBU, given i failed components, (denoted by i-NBU) if either

(a) $P(E_i \cap \{T > t\}) = 0 \; \forall t$ or

(b) $P(E_i \cap \{T > t\}) > 0 \; \forall t$ and

$$P(T > x) \geq P(T > x + t \mid E_i \cap \{T > t\}) \; \forall x > 0 \text{ and } \forall t \in S_F.$$

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Note: Under condition (a) above, the system fails with probability 1 if i of its components have failed (that is, if its signature $\mathbf{s}$ satisfies $\sum_{j=1}^{i} s_j = 1$), so the new system is clearly better that the "used system" with i failed components. Under condition (b), the system has a positive probability of survival, given i failed components, but the residual lifetime of the used system is stochastically smaller than the lifetime of a new system.

IV. Dynamic Versions of Aging

A more restricted but very interesting version of the NBU concept is that of systems that are uniformly NBU. We define this property as follows.

Defn. 5: An n-component system is said to be Uniformly New Better than Used (UNBU) if it is i-NBU for $i = 0, 1, 2, \ldots, n$.

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Note: Since the conditional probability $P(T > x + t \mid T > t)$ may be written as the mixture

$$P(T > x + t \mid T > t) = \sum_{i \in A} P(T > x + t \mid E_i \cap \{T > t\}) P(E_i \mid T > t),$$

where $A = \{j \mid P(E_j \cap \{T > t\}) > 0\}$, it follows that if a system is UNBU, then

$$P(T > x + t \mid T > t) \leq \sum_{i \in A} P(T > x) P(E_i \mid T > t) = P(T > x),$$

that is, the system is NBU, as expected.

IV. Dynamic Versions of Aging

Some Useful Notation:

Consider a mixed system based on m components with i.i.d. lifetimes. We will henceforth employ more precise notation for the signature of such a system and for the signature of the equivalent mixed system of order $n \geq m$:

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The signature of the equivalent system of order n is $\mathbf{s}_n(n-i)$.

IV. Dynamic Versions of Aging

Using the notation above:

If $\overline{F}$ is the reliability function of a new component, and $\overline{G}(x \mid t) = \overline{F}(x+t)/\overline{F}(t)$, then the residual reliability function of working but used system with i failed components at time t may be written either as

$$\overline{G}_T(x \mid t, i) = \sum_{j=1}^{n-i} s_{n-i,j}(n-i)\overline{G}_{j:n-i}(x) \quad \text{for } x > 0,$$

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where $\overline{G}_{j:n-i}(x)$ is the reliability function of the jth order statistic from a random sample of size $n-i$ from $\overline{G}(x \mid t)$ or, alternatively, as

$$\overline{G}_T(x \mid t, i) = \sum_{j=1}^{n} s_{n,j}(n-i)\overline{G}_{j:n}(t) \quad \text{for } x > 0,$$

where $\overline{G}_{j:n}(t)$ is the reliability function of the jth order statistic from a random sample of size n from $\overline{G}(x \mid t)$.

IV. Dynamic Versions of Aging

The following example illustrates the UNBU property.

Example 3. Consider a 4-component mixed system with signature $\mathbf{s}(4) = (.1, .2, .3, .4)$. Assume all components have **i.i.d. exponential lifetimes**. The dynamic signatures for $i = 1, 2$ and 3 are

$$\mathbf{s}(3) = (2/9, 3/9, 4/9) = (3/18, 4/18, 5/18, 6/18) = \mathbf{s}_4(3),$$

$$\mathbf{s}(2) = (3/7, 4/7) = (9/42, 10/42, 11/42, 12/42) = \mathbf{s}_4(2),$$

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and

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Since $\mathbf{s}(4)$ is stochastically larger than all three of the dynamic signatures (of order 4), it follows that the system with signature $\mathbf{s}(4)$ is UNBU when components have i.i.d. exponential lifetimes.

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Similar results can be obtained for the complimentary notions of conditional and uniformly New Worse than Used systems.

IV. Dynamic Versions of Aging

Lemma 1: Suppose that $F_1 \geq_{\text{st}} F_2$. Assume that $U_1, \ldots, U_n \overset{\text{i.i.d.}}{\sim} F_1$ and that $V_1, \ldots, V_n \overset{\text{i.i.d.}}{\sim} F_2$. Then $U_{j:n} \geq_{\text{st}} V_{j:n}$ for $j = 1, \ldots, n$.

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Using the lemma above, we may infer from the NBU assumption on F that $F_{i:n} \geq_{\text{st}} G_{t,i:n}$. The following result gives sufficient conditions for a system to be UNBU.

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Using the lemma above, we may infer from the NBU assumption on F that $F_{i:n} \geq_{\text{st}} G_{t,i:n}$. The following result gives sufficient conditions for a system to be UNBU.

Theorem 4: Let $\mathbf{s}_n(n)$ be the signature, and T the lifetime, of a mixed system based on n components whose lifetimes are i.i.d. $\sim F$. Assume that F is NBU and that

$$\mathbf{s}_n(n) \geq_{\text{st}} \mathbf{s}_n(n-i) \text{ for } i = 1, \ldots, n. \tag{1}$$

For arbitrary $x, t > 0$, and for $i \in \{1, \ldots, n\} \ni \; P(E_i \cap \{T > t\}) > 0$, define

$$\overline{G}_T(x \mid t, i) = P(T > x + t \mid E_i \cap \{T > t\}).$$

Then for all such i,

$$\overline{F}_T(x) \geq \overline{G}_T(x \mid t, i), \tag{2}$$

and the system is UNBU.

IV. Dynamic Versions of Aging

Proof of Theorem 4: First, we note that (2) holds for $i = 0$ by virtue of the NBU assumption on F. For $i > 0$ such that $P(\{X_{i:n} \leq t < X_{i+1:n}\} \cap \{T > t\}) > 0$, we see that the system is i-NBU as follows:

$$\begin{aligned}
\overline{F}_T(x) &= \sum_{j=1}^{n} s_{n,j}(n)\overline{F}_{j:n}(x) && \text{(by Theorem 1)} \\
&\geq \sum_{j=1}^{n} s_{n,j}(n)\overline{G}_{t,j:n}(x) && \text{(by the NBU assumption and Lemma 1)} \\
&\geq \sum_{j=1}^{n} s_{n,j}(n-i)\overline{G}_{t,j:n}(x) && \text{(by assumption (1))} \\
&= \overline{G}_T(x \mid t, i) && \text{(by Theorem 1).}
\end{aligned}$$

For $i > 0$ such that $P(\{X_{i:n} \leq t < X_{i+1:n}\} \cap \{T > t\}) = 0$, the system is i-NBU by Definition 4(a). Thus, the system is i-NBU for all i and hence is UNBU. ■

V. On the Engineering Practice of "Burn In".

A. Optimal Burn in for systems whose components have i.i.d. exponential lifetimes.

Example 1. Assume i.i.d. exponential component lifetimes. Consider the 3-component system with signature vector $\mathbf{s} = (11/16, 4/16, 1/16)$. Then

$$\mathbf{s}_2(2) = \left(\frac{4}{5}, \frac{1}{5}\right) = \left(\frac{8}{15}, \frac{1}{3}, \frac{2}{15}\right) = \mathbf{s}_3(2)$$

and

$$\mathbf{s}_1(1) = (1) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \mathbf{s}_3(1).$$

Clearly, $\mathbf{s}_3(3) \leq_{st} \mathbf{s}_3(2)$ and $\mathbf{s}_3(3) \leq_{st} \mathbf{s}_3(1)$, which proves that the 3-component system with signature $(11/16, 4/16, 1/16)$ performs better when burned in to the first or to the second component failure, provided of course that the system is still working after these burn-in procedures.

V. On the Engineering Practice of "Burn In".

Cost Considerations

Our approach will require the selection of a particular class of criterion functions by which different burn-in strategies can be compared. The criterion to be used is a form of "performance per unit cost"; define

$$m(a, b) = \text{Expected Lifetime} \;/\; \text{Expected Cost} \; = ET/(a + b),$$

where a is the fixed cost of the system being employed and b is the expected or real cost of the components needed to field a working (new or burned-in) system.

V. On the Engineering Practice of “Burn In”.

Example 1. (cont'd): We are dealing with the three choices – a new system or a working system that's been burned-in to either the first or the second component failure. Let the residual lifetimes of these three systems be denoted by T_3, T_2 and T_1, the subscript representing the number of working components when the system is sold or deployed. If component lifetimes are i.i.d. Exp(1), we find, using the identity

$$ET = \sum_{i=1}^{3} s_i EX_{i:3},$$

that $ET_3 = 53/96$, $ET_2 = 63/90$ and $ET_1 = 1$.

V. On the Engineering Practice of "Burn In".

We now account for costs. The fixed costs of the 3-component system (with signature $\left(\frac{11}{16}, \frac{4}{16}, \frac{1}{16}\right)$) under consideration is "a." We assume, again, that component lifetimes are i.i.d. Exp(1). Denoting the cost of an individual component by "c," the cost of the components associated with fielding a new system is simply

$$b = 3c.$$

The number of trials needed to obtain a working system with one failed component is a geometric variable with $p = 5/16$ and expected value $16/5$. Thus, the expected component cost to field a working system that has been burned in to $X_{1:3}$ is

$$b = 2c + 16c/5 = 5.2c.$$

Proceeding similarly, the expected component cost incurred by burn in to $X_{2:3}$ is

$$b = c + 16(1.2667c) = 21.2667c.$$

V. On the Engineering Practice of "Burn In".

Using a "performance per unit cost" criterion, fielding a system burned in to $X_{1:3}$ is to be preferred to the use of a new system if

$$0.7/(a + 5.2c) > 0.5521/(a + 3c),$$

which is equivalent to the inequality $c < 0.1918a$. We conclude that fielding a system that has been successfully burned in to the first component failure is best if your components aren't too expensive.

V. On the Engineering Practice of "Burn In".

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Similarly, burn-in to two component failures is preferable to burn-in to one component failure if

$$1/(a + 21.2667c) > 0.7/(a + 5.2c),$$

that is, if $c < 0.031a$.

V. On the Engineering Practice of "Burn In".

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We thus find that the intervals $(0, 0.031a)$, $(0.031a, 0.1918a)$ and $(0.1918a, \infty)$ determine the range of the component cost c corresponding to the overall optimality of the use of 2-component burn-in, 1-component burn-in or the new system, respectively.

V. On the Engineering Practice of "Burn In".

B. Optimal Burn in for systems whose components have i.i.d. non-exponential lifetimes.

Theorem 14: (Balakrishnan, Navarro and Samaniego (2008)) Suppose that the component lifetimes of a system of order n are given as $X_1, X_2, \ldots, X_n \overset{iid}{\sim} F$, let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the corresponding order statistics and let T be the system's lifetime. Assume that the system has been successfully burned in to $X_{k:n}$, that is, assume that $T > X_{k:n}$. If $Z_1, \ldots, Z_{n-k}$ are the lifetimes of the $n-k$ surviving components and $Z_{1:n-k}, \ldots, Z_{n-k:n-k}$ are the corresponding order statistics, then for arbitrary $i \in \{1, 2, \ldots, n-k\}$,

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(a) the distribution of Z_i, given $T > X_{k:n}$, is the equiprobable mixture of the ordered lifetimes of the $n-k$ surviving components, i.e., $Z_i = X_{k+j:n}$ with probability $\frac{1}{n-k}$ for $j = 1, \ldots, n-k$,

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(b) $Z_{i:n-k} \overset{d}{=} X_{k+i:n}$.

V. On the Engineering Practice of "Burn In".

We will now apply the result above to the comparison of burn-in strategies for systems with nonexponential component lifetimes. Let $W(\alpha, \beta)$ be the Weibull distribution with shape parameter α and scale parameter β. We consider the DFR and the IFR cases in turn.

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Example 2: $W(0.5, 1)$ Component Lifetimes

Consider, again, the 3-component system with signature $(11/16, 4/16, 1/16)$. The expected order statistics of $W(0.5, 1)$ are 0.22222, 1.05556 and 4.72222 respectively. From Theorem 11(b), we obtain the expected residual lifetime of the system to be

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$ET_1 - EX_{2:3} = 4.7222 - 1.0556 = 3.6667$ (for a system burned in to $X_{2:3}$)

V. On the Engineering Practice of "Burn In".

Using a "performance per unit cost" criterion, burn-in to the first component failure is to be preferred to the use of a new system if

$$1.5666/(a + 5.2c) > .7118/(a + 3c),$$

an inequality that holds for all $c \in (0, \infty)$. It follows that when component lifetimes have this particular DFR distribution, burning this system to the first component failure will always improve upon a new system.

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an inequality that holds for all $c \in (0, \infty)$. It follows that when component lifetimes have this particular DFR distribution, burning this system to the first component failure will always improve upon a new system.

Burn in to $X_{2:3}$ will be better that burn in to $X_{1:3}$ if

$$3.6667/(a + 21.2667) > 1.5666/(a + 5.2c),$$

That is, if $c < 0.1474a$.

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That is, if $c < 0.1474a$.

Thus, for components with $W(0.5, 1)$ lifetimes, *burn in to* $X_{2:3}$ is to be preferred if $c \in (0, 0.1474a)$ and *burn in to* $X_{1:3}$ is preferred if $c \in (0.1474a, \infty)$.

V. On the Engineering Practice of "Burn In".

Example 3: $W(2,1)$ Component Lifetimes

The expected order statistics of $W(2,1)$ are 0.51166, 0.85664 and 1.29037, respectively. For the 3-component system with signature $(11/16, 4/16, 1/16)$, we obtain from Theorem 11(b) that the expected residual lifetime of the system is

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$ET_1 - EX_{2:3} = 1.2904 - 0.8566 = 0.4337$ (for a system burned in to $X_{2:3}$)

V. On the Engineering Practice of "Burn In".

Using a "performance per unit cost" criterion, burn-in to the first component failure is to be preferred to the use of a new system if

$$0.4272/(a + 5.2c) > 0.6466/(a + 3c). \quad (3)$$

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Using a "performance per unit cost" criterion, burn-in to the first component failure is to be preferred to the use of a new system if

$$0.4272/(a + 5.2c) > 0.6466/(a + 3c). \tag{3}$$

Further, burn in to $X_{2:3}$ is preferred to the use of a new system if

$$0.4337/(a + 21.2667) > 0.6466/(a + 5.2c). \tag{4}$$

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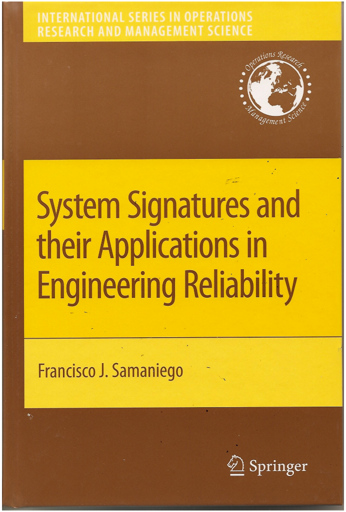
Since both (2) and (3) fail for all c, fielding a new system is superior to burning the system in when component lifetimes are $W(2, 1)$.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