Two observations on set partitions and pattern-avoiding permutations

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PP 2011, Cal Poly University, San Luis Obispo
June 24, 2011
Talk outline

Set partitions of \([n]\) are equinumerous with \(S_n(\tau)\), where \(\tau\) is any one of 1-23, 24\(\bar{1}\)3, and their equivalents under inverse, reverse, and complement.

- The barred pattern helps clarify and slightly generalize a result of Rodica Simion.

- Several familiar counting sequences arise for \(S_n(B)\) where \(B\) is a subset of the above patterns.
Canonical forms for set partitions

<table>
<thead>
<tr>
<th>Form</th>
<th>each block</th>
<th>min entries</th>
<th>set partition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>↑</td>
<td>↑</td>
<td>13 258 469 7</td>
</tr>
<tr>
<td>2</td>
<td>↑</td>
<td>↓</td>
<td>7 469 258 13</td>
</tr>
<tr>
<td>3</td>
<td>smallest first, then ↓</td>
<td>↓</td>
<td>7 496 285 13</td>
</tr>
</tbody>
</table>

“Write in Form 2 and concatenate blocks” is a bijection

\[ P([n]) \rightarrow S_n(24\overline{1}3), \]

and, “Write in Form 3 and concatenate blocks” is a bijection [Claesson, 2001]

\[ P([n]) \rightarrow S_n(1-23). \]
Restricted growth sequence $(a_i)_{i=1}^{n}$:

$$a_1 = 1, \quad a_i \leq \max_{1 \leq j < i} \{a_j\} + 1.$$ 

Setting “$a_i =$ position of block containing $i$ using Form 1” is a well known bijection $\mathcal{P}([n]) \rightarrow \mathcal{RGS}[n]$.

The statistics $ls$ and $rb$ on set partitions.

$$ls(1 \ 3 \ / \ 2 \ 5 \ 8 \ / \ 4 \ 6 \ 9 \ / \ 7) =$$

$$ls(1 \ 2 \ 1 \ 3 \ 2 \ 3 \ 4 \ 2 \ 3) =$$

$$0 + 1 + 0 + 2 + 1 + 2 + 3 + 1 + 2 = 12$$

$$rb = 3 + 2 + 3 + 1 + 2 + 1 + 0 + 1 + 0 = 13$$

**Theorem 1** (Simion, 1994). The statistics $ls$ and $rb$ have a symmetric joint distribution on noncrossing (NC) partitions of $[n]$. 
Proposition 2 (S). For all partitions $\pi$, $\text{ls}(\pi) = \text{Sum(block-size vector)}$

<table>
<thead>
<tr>
<th>Form 2</th>
<th>7</th>
<th>4</th>
<th>6</th>
<th>9</th>
<th>2</th>
<th>5</th>
<th>8</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>block sizes</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>block-size vector</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>7</td>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Proposition 3 (S). For NC partitions $\pi$, $\text{rb}(\pi) = \text{Sum(min-entry vector)}$

<table>
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<tr>
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<th>6</th>
<th>9</th>
<th>2</th>
<th>5</th>
<th>8</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>min entries</td>
<td>7</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>min-entry vector</td>
<td>6</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>
**Theorem 2** (S). *The statistics block-size vector and min-entry vector have a symmetric joint distribution on noncrossing partitions of \([n]\).*

Proof. Translate everything from partitions (written in Form 2) to 2413-avoiding permutations.

- **min-entry vector** → LR minima − 1
- **block-size vector** → locations(LR minima) − 1
- **NC Partitions** → 243- (or 132-) avoiding permutations

Inverse permutation is a map on \(S_n\) that interchanges LR minima and locations(LR minima), and preserves \(S_n(132)\).
The inverse sends $S_n(24\bar{1}3)$ to $S_n(\bar{3}142)$, and so preserves $S_n(24\bar{1}3, \bar{3}142)$.

We can conclude that $\text{Sum(min-entry vector)}$ and $\text{Sum(block-size vector)}$—though not $\text{ls}$ and $\text{rb}$—have a symmetric joint distribution on $S_n(24\bar{1}3, \bar{3}142)$. How many such permutations?

A permutation $\pi$ avoids both $24\bar{1}3$ and $\bar{3}142$ if and only if $\pi$ and $\pi^{-1}$ both avoid $\bar{3}142$, also [Pudwell]

$\Longleftrightarrow \pi$ avoids $\bar{4}25\bar{1}3$. 
**Theorem 4** (Bousquet-Mélou et al., 2010).

\[
| \{ \pi \in S_n : \pi \text{ and } \pi^{-1} \text{ both avoid } 3142 \} | = \sum_{k \geq 0} \binom{k}{2} + n - 1
\]

**Count 5.** *This sum also counts the set \( \{ \pi \in S_n : \pi \text{ and } \pi^{-1} \text{ both avoid } 1-23 \} \).*

The **LR min decomposition** of a permutation:

\[
6 9 8 4 5 7 2 1 4.
\]

Turn the LR minima into dividers

\[
\square 9 8 \square 5 7 \square \square 4.
\]

to get the **LR min segments**

\[
9 8 5 7 \epsilon 4.
\]

**DeleteLRMin**\( (\pi) \): \( 9 8 5 7 4 \).

5 - 4 is a **small drop**
For any permutation $p$,

\[
\begin{align*}
\text{\textbf{\textit{p}} avoids 1-23} & \iff \text{each of its LR min segments is } \downarrow. \\
\text{\textbf{\textit{p}^{-1}}} \text{ avoids 1-23} & \iff \text{\textbf{\textit{p}}} \text{ avoids } u-v-(v+1) \iff \\
\text{DeleteLRMin}(\text{\textbf{\textit{p}}}) & \text{ contains no small jumps. }
\end{align*}
\]

\[
\text{\textbf{\textit{p}}} = 6 \ 8 \ 3 \ 5 \ 7 \ 1 \ 2 \ 4.
\]

\[
\begin{align*}
\text{\textbf{\textit{p}}} \text{ avoids } 24\bar{1}\bar{3} & \iff \text{each of its LR min segments is } \uparrow. \\
\text{\textbf{\textit{p}^{-1}}} \text{ avoids } 24\bar{1}\bar{3} & \iff \text{\textbf{\textit{p}}} \text{ avoids } \bar{3}142 \iff \\
\text{DeleteLRMin}(\text{\textbf{\textit{p}}}) & \text{ contains no small drops. }
\end{align*}
\]
Proof of Count 5. Given a permutation where the segments ↓ and DeleteLRMin contains no small jumps,

1. Connect each small drop by an arc

```
8 4 10 2 7 6 3 1 9 5
```

segments ↓, no small jumps

2. Reverse each segment

```
8 4 10 2 3 6 7 1 5 9
```

segments ↑, some small drops

3. For each run of connected arcs, rearrange its entries in ↑ order.

```
8 4 9 2 3 5 6 1 7 10
```

segments ↑, no small drops
Count 6.

\[ | S_n(3142, 1-23) | = \sum_{k \geq 0} \binom{k+1}{2} \binom{n-k}{n-k} . \]

Proof. We wish to construct permutations where DeleteLRMin contains no small drops and each segment is \( \downarrow \). First, list the 2-element subsets of 1, 2, \ldots, \( k + 1 \), as columns. 
\( (k \) will be \#LR minima = \#segments in the constructed permutation.)
Select \( n - k \) of these columns, arranged in reverse order (here \( k = 4 \), and let's take \( n = 10 \)).

The selected columns determine the locations and values of the LR min segment entries.

\[
\begin{array}{ccccccc}
4 & 2 & 2 & 2 & 1 & 1 & \leftarrow \text{segment locations (from right)} \\
5 & 5 & 4 & 3 & 4 & 2 & \leftarrow \text{determines segment entries} \\
4 & 5 & 2 & 1 & 3 & 0 & \leftarrow \text{increments} \\
9 & 10 & 6 & 4 & 7 & 2 & \leftarrow \text{actual segment entries} \\
\end{array}
\]

\[8 \ 9 \ 5 \ 3 \ 10 \ 6 \ 4 \ 1 \ 7 \ 2 \in S_n(\overline{3142}, \ 1-23) \]
A set partition with all blocks of size 1 or 2 \( \sim \) a permutation that is an involution \( \sim \) a partial matching.

Represent as 2-row array, top row decreasing:

\[
\begin{array}{ccccccc}
10 & 7 & 6 & 4 & 3 & 2 & 1 \\
9 & 11 & 8 & 5 \\
\end{array}
\]

\( \ast \ 9 \ \ast \ 8 \ \ast \) is a \textit{small drop} in bottom row

\begin{lemma}
The number of partial matchings of \([n]\) with \(k\) columns and, in the bottom row,
\begin{enumerate}[(i)]
    \item no small drops is \( \binom{k+1}{2k+1-n} \),
    \item no small drops and no small jumps is \( \left\{ \binom{k+1}{2k+1-n} \right\} \).
\end{enumerate}
\end{lemma}

\begin{proof}
In each case, deleting the last column produces another such matching. This observation leads to a recursive construction and a Stirling-like recurrence with solution as stated.
\end{proof}
Count 8.

\[ |S_n(\overline{3}142, \overline{3}142^{-1}, 1-23)| = \sum_{k \geq 0} \left[ \begin{array}{c} k + 1 \\ 2k + 1 - n \end{array} \right]. \]

Proof. Recall 1-23 \(\leftrightarrow\) segments \(\downarrow\), \(\overline{3}142 \leftrightarrow\) no small drops, and \(\overline{3}142^{-1} = 24 \overline{1}3 \leftrightarrow\) segments \(\uparrow\).

Avoiding all three patterns requires that each segment is empty or a singleton, and so a permutation avoids all three patterns \(\leftrightarrow\) its LR min decomposition forms a partial matching on \([n]\) with no small drops in the second row. Apply Lemma 1(i). \(\square\)

Remark. A refined count by first entry, \(j\), gives a combinatorial interpretation of the identity

\[ \sum_{j \geq 0} \binom{2k - j}{n - j} \left[ \begin{array}{c} k \\ 2k - j \end{array} \right] = \left[ \begin{array}{c} k + 1 \\ 2k + 1 - n \end{array} \right]. \]
Count 9.

\[ |S_n(\overline{3142}, \overline{3142}^{-1}, 1-23, 1-23^{-1})| = \sum_{k \geq 0} \left\{ \frac{k + 1}{2k + 1 - n} \right\}. \]

Proof. A permutation avoids all 4 patterns ⇔ its LR min decomposition forms a partial matching on \([n]\) with no small drops and no small jumps in the second row. Apply Lemma 1(ii). \qed
Conjecture 10. \( S_n(\overline{3142}, \overline{3142}^{-1}, 12-3) \) is counted by 1, 1, 2, 4, 8, 17, 37, 82, ..., sequence A004148 in OEIS.

The manifestations of this sequence include “secondary structures of RNA molecules” and peak-less Motzkin paths.