DAVID CALLAN
Department of Statistics
University of Wisconsin-Madison
1300 University Ave
Madison, WI 53706-1532
callan@stat.wisc.edu

1567. *Proposed by Dennis Spellman, Philadelphia, PA and William P. Wardlaw, United States Naval Academy, Annapolis, MD.*

Find the Smith normal form over the integers of the $n \times n$ matrix A with entries $a_{ij} = j^i$.

SOLUTION I

The kth determinantal divisor d_k of an integer matrix A, $1 \leq k \leq \text{rank}(A)$, is defined to be the greatest common divisor of all k by k minors of A. Considering cofactor expansions, it is clear that d_{k-1} divides d_k for $1 \leq k \leq \text{rank}(A)$ (with $d_0 := 1$). Furthermore, the ratio d_k/d_{k-1} is known to be the kth diagonal entry of the Smith normal form of A. For the present matrix, all minors are determinants of Vandermonde matrices and so it is easy to see that $\text{rank}(A) = n$ and d_k is the upper left k by k minor of A, namely $k!(k-1)!(k-2)!\ldots 1!$. The Smith normal form of A thus has $k!$ as its kth diagonal entry, $1 \leq k \leq n$.

SOLUTION II

Let $\left[{i \atop j} \right]$ and $\left\{ {i \atop j} \right\}$ denote the Stirling cycle and subset numbers (a. k. a. first and second kind). We claim that the diagonal matrix D with diagonal entries $d_{ii} = i!$, $1 \leq i \leq n$, is the Smith normal form of the given matrix A; furthermore, unimodular (= integer with determinant ± 1) matrices U, V such that $UAV = D$ are given by $u_{ij} = (-1)^{i-j} \left[{i \atop j} \right]$ and $v_{ij} = (-1)^{j-i}\binom{j}{i}$. It is well known that $(V^{-1})_{ij} = \binom{j}{i}$, and—at least since Donald Knuth's May 1992 Monthly article, *Two Notes on Notation*—that $(U^{-1})_{ij} = \left\{ {i \atop j} \right\}$. This establishes unimodularity of U and V. Now the (i,j)-entry of $U^{-1}D$ is $\left\{ {i \atop j} \right\} j!$—the number of surjective functions from an i-element set to a j-element set. By the inclusion-exclusion principle, this number is $\sum_k (-1)^k \binom{j}{k}(j-k)^i$, which is the (i,j)-entry of AV. Thus $U^{-1}D = AV$ and the claim follows.