that there are infinitely many positive integers \( k \) such that
\[
\sum_{n=1}^{\infty} c_n a_n^k \neq 0.
\]

**Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands.** We may assume that \( a_n \neq 0 \) for all \( n \). If \( \sum_{n=1}^{\infty} c_n a_n^k = 0 \) for all sufficiently large \( k \), then the analytic function
\[
F(z) = \sum_{k=1}^{\infty} \left( \sum_{n=1}^{\infty} c_n a_n^k \right) z^k
\]
is a polynomial. On the other hand, in a neighborhood of \( z = 0 \) we have
\[
F(z) = \sum_{n=1}^{\infty} c_n \left( \sum_{k=1}^{\infty} (a_n z)^k \right) = \sum_{n=1}^{\infty} c_n \frac{a_n z}{1 - a_n z},
\]
which is a function that has poles at \( z = a_n^{-1} \). This is a contradiction.

*Editorial comment.* The example \( a_n = c_n = 2^{-n} \) shows that the sequence
\[
\left\{ \sum_{n=1}^{\infty} c_n a_n^k \right\}_{k=1}^{\infty}
\]
can converge to zero. Per Yuan Wu contributed the references [1] and [2] below. Lenard’s paper gives an example showing that the assumption that \( \sum |a_n| \) converges is essential. Weber’s paper contains a solution of the case \( c_n = 1 \), rephrased as an operator-theoretic problem.

**REFERENCES**


Solved also by U. Abel (Germany), David Borwein (Canada), J. P. Grivaux (France), J. H. Lindsey II, T. S. Norfolk, A. Riese, the late David Richman, R. Stong, P. Tracy, P. Y. Wu (China), University of South Alabama Problem Group, and the proposer.

**POSTMORTEM COMMENTS ON ELEMENTARY PROBLEMS**

**Inequalities Involving Trigonometric Functions**


(a) Prove that if \( 0 < x < \pi/2 \), then
\[
\left( \frac{\sin x}{x} \right)^2 + \frac{\tan x}{x} > 2.
\]
(b) Find the largest constant $c$ such that

\[
\frac{\sin x}{x}^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x
\]

for $0 < x < \pi/2$.

Editorial remark. The editors overlooked the fact that the following solution of (b) is much shorter than the published solution by Jean Anglesio. Other solutions of (b) were received from Richard A. Groeneveld, Kee-Wai Lau (Hong Kong), J. Ernest Wilkins, Jr., and the Lamar University Problem Solving Group.

Solution II of (b) by David Callan, University of Wisconsin, Madison. We must determine the infimum on the interval $(0, \pi/2)$ of the function $\phi$ defined on the interval $(0, \pi)$ by

\[
(\ast) \quad \phi(x) = \left\{ (x^{-1} \sin x)^2 + x^{-1} \tan x - 2 \right\} / \{x^3 \tan x\}
\]

\[
= 2^{-1} x^{-5} \sin 2x + x^{-4} - 2x^{-3} \cot x.
\]

We claim that $\phi$ is decreasing on $(0, \pi/2)$, so that

\[
\inf_{0 < x < \pi/2} \phi(x) = \phi(\pi/2) = 16/\pi^4.
\]

Thus $16/\pi^4$ is the largest value of $c$ satisfying (b).

On the interval $(0, \pi)$ we have the two Maclaurin expansions

\[
\sin 2x = 2x - 4x^3/3 + \sum_{n=0}^{\infty} (-1)^n (2x)^{2n+5}/(2n + 5)!,
\]

\[
x \cot x = 1 - x^2/3 - \sum_{n=0}^{\infty} 2^{2n+4} |B_{2n+4}| x^{2n+4} / (2n + 4)!
\]

the latter is easily deduced from the fact that the Bernoulli numbers $B_r$ can be defined by the power series expansion $t/(e^t - 1) = \Sigma_{r=0}^{\infty} B_r t^r/r!$ for $|t| < 2\pi$. Hence for $0 < x < \pi$ we have

\[
\phi(x) = \sum_{n=0}^{\infty} \frac{2^{2n+5}}{(2n + 4)!} \left\{ |B_{2n+4}| + \frac{(-1)^n}{2(2n + 5)} \right\} x^{2n}.
\]

Thus, if $\psi(x) = \phi(\sqrt{x})$ for $0 < x < \pi^2$, we have

\[
\psi(x) = \frac{8}{45} - \frac{8x}{945} + \frac{16x^2}{14175} + \sum_{n=3}^{\infty} \frac{2^{2n+5}}{(2n + 4)!} \left\{ |B_{2n+4}| + \frac{(-1)^n}{4n + 10} \right\} x^{n}.
\]

Since

\[
2(2n + 5) |B_{2n+4}| = 4(2n + 5)! (2\pi)^{-2n-4} \sum_{k=1}^{\infty} k^{-2n-4} > 4(2n + 5)! (2\pi)^{-2n-4} > 1
\]

for $n > 3$, we have $|B_{2n+4}| > 1/(4n + 10)$ for $n > 3$. Thus $\psi''(x) > 0$ on the interval $(0, \pi^2)$ and so $\psi'$ is increasing there. On the other hand, from (\ast) we
readily obtain
\[ \psi'(\pi/2)^2 = \pi^{-1}\psi'(\pi/2) = (2/\pi)^2(1 - 10/\pi^2) < 0. \]

It follows that \( \psi(x) \) is negative throughout the interval \( (0, (\pi/2)^2) \) and so \( \psi \) is decreasing there. Hence \( \phi \) is decreasing on the interval \( (0, \pi/2) \) and our claimed result is established.

**Rationally Independent Subsets of the Reals**


Suppose \( S \) is a set of \( 2n + 1 \) irrational real numbers. Prove that \( S \) has a subset \( T \) of cardinality \( n + 1 \) such that no nonempty subset of \( T \) has a rational sum.

**Editorial remark.** Professor Günter Rote of the Technical University of Graz, Austria, has pointed out to us that the second solution published in the November, 1990 issue is flawed, in that the hyperplane mentioned in the solution should contain the set \( \mathbb{Q} \) of rational numbers rather than merely the origin 0. A corrected version of this solution is as follows.

**Solution II by an anonymous contributor.** Let \( V \) be the finite-dimensional vector space over \( \mathbb{Q} \) generated by 1 and the elements of \( S \). Since \( S \) is a finite set of real numbers none of which is contained in \( \mathbb{Q} \), there exists a hyperplane \( H \) through the one-dimensional subspace of \( V \) generated by 1 such that \( H \) contains no elements of \( S \). Clearly \( n + 1 \) of the elements of \( S \) fall on the same side of \( H \). These \( n + 1 \) elements of \( S \) form a set \( T \) having the property stated in the problem. In fact no nontrivial linear combination of the elements of \( T \) with nonnegative rational coefficients is in \( H \), so that in particular no non-empty subset of \( T \) has a rational sum.

**Sum-Free Sets Modulo \( n \)**

E 3346 [1989, 735; 1991, 368]. *Proposed by Dean S. Clark, University of Rhode Island, Kingston.*

Call a subset \( T \) of \( \mathbb{Z} \mod n \) sum-free if the sum of two distinct elements of \( T \) is not in \( T \). Let \( s(n) \) be the maximum cardinality of a sum-free subset of \( \mathbb{Z} \mod n \).

(a) Prove that \( s(n) = n/2 \) if \( n \) is even.

(b) If \( n \) is odd, prove that \( s(n) \geq \lfloor n/3 \rfloor + 1 \).

(c)* If \( n \) is odd, is \( s(n) = \lfloor n/3 \rfloor + 1 \)?

**Editorial Comment.** For odd \( n \) several solvers (correctly) conjectured that “\( s(n) = \lfloor n/3 \rfloor + 1 \) if \( n \) has no prime factors congruent to 5 modulo 6, but \( s(n) = n(p + 1)/(3p) \) if \( p \) is the smallest prime factor of \( n \) congruent to 5 modulo 6.”

Fred Galvin has remarked to us that the preceding conjecture follows easily from the following known result: If \( s^*(n) \) is the maximum cardinality of a subset \( T \) of \( \mathbb{Z} \mod n \) which does not contain the sum of any two elements of \( T \), distinct or not, then

\[
(*) \quad s^*(n) = \max_{d | n} \left\{ \left\lfloor \frac{d + 1}{3} \right\rfloor \frac{n}{d} \right\}.
\]