Cesàro’s Integral Formula for the Bell Numbers (Corrected)

DAVID CALLAN
Department of Statistics
University of Wisconsin-Madison
Medical Science Center
1300 University Ave
Madison, WI 53706-1532
callan@stat.wisc.edu

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In 1885, Cesàro [1] gave the remarkable formula

\[ N_p = \frac{2}{\pi e} \int_0^\pi e^{e^{\cos \theta \cos (\sin \theta)}} \sin( e^{\cos \theta \sin (\sin \theta)}) \sin p\theta \ d\theta \]

where \( (N_p)_{p \geq 1} = (1, 2, 5, 15, 52, 203, \ldots) \) are the modern-day Bell numbers. This formula was reproduced verbatim in the Editorial Comment on a 1941 Monthly problem [2] (the notation \( N_p \) for Bell number was still in use then). I have not seen it in recent works and, while it’s not very profound, I think it deserves to be better known.

Unfortunately, it contains a typographical error: a factor of \( p! \) is omitted. The correct formula, with \( n \) in place of \( p \) and using \( B_n \) for Bell number, is

\[ B_n = \frac{2}{\pi e} n! \int_0^\pi e^{e^{\cos \theta \cos (\sin \theta)}} \sin( e^{\cos \theta \sin (\sin \theta)}) \sin n\theta \ d\theta \quad n \geq 1. \]

The integrand is the imaginary part of \( e^{e^{i\theta}} \sin n\theta \), and so an equivalent formula is

\[ B_n = \frac{2}{\pi e} n! \text{Im} \left( \int_0^\pi e^{e^{i\theta}} \sin n\theta \ d\theta \right). \quad (1) \]

The formula (1) is quite simple to prove modulo a few standard facts about set partitions. Recall that the Stirling partition number \( \{n\}_k \) is the number of partitions of \( [n] = \{1, 2, \ldots, n\} \) into \( k \) nonempty blocks and the Bell number \( B_n = \sum_{k=1}^n \{n\}_k \) counts all partitions of \( [n] \). Thus \( k!\{n\}_k \) counts ordered partitions of \( [n] \) into \( k \) blocks (the \( k! \) factor serves to order the blocks) or, equivalently, counts surjective functions \( f \) from \( [n] \)}
onto \([k]\) (the \(j\)th block is \(f^{-1}(j)\)). Since the number of unrestricted functions from \([n]\) to \([j]\) is \(j^n\), a classic application of the inclusion-exclusion principle yields

\[
k! \binom{n}{k} = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n. \tag{2}
\]

The trig identity underlying Cesàro’s formula is nothing more than the orthogonality of sines on \([0, \pi]\):

\[
\int_0^\pi \sin m\theta \sin n\theta \, d\theta = \begin{cases} \frac{\pi}{2} & \text{if } m = n, \\ 0 & \text{if } m \neq n \end{cases}
\]

for \(m, n\) nonnegative integers. Using the Taylor expansion \(e^x = \sum_{m \geq 0} \frac{x^m}{m!}\) and DeMoivre’s formula \(e^{i\theta} = \cos \theta + i \sin \theta\), it follows that

\[
\text{Im} \left( \int_0^\pi e^{je^{i\theta}} \sin n\theta \, d\theta \right) = \frac{j^n \pi}{n!} \frac{\pi}{2} \tag{3}
\]

for integer \(j \geq 0\). Now we show that

\[
\text{Im} \left( \int_0^\pi \left( \frac{e^{je^{i\theta}} - 1}{k!} \right) \sin n\theta \, d\theta \right) = \frac{1}{n!} \binom{n}{k} \frac{\pi}{2} \tag{4}
\]

for integer \(k \geq 0\) (of course, \(\binom{n}{k} = 0\) for \(n > k = 0\) and for \(k > n\)).

**Proof**  The binomial theorem implies the left hand side is

\[
\frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \text{Im} \left( \int_0^\pi e^{je^{i\theta}} \sin n\theta \, d\theta \right) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n \frac{\pi}{n!} \frac{\pi}{2} = \frac{1}{n!} \binom{n}{k} \frac{\pi}{2} \tag{3} = \frac{1}{n!} \binom{n}{k} \frac{\pi}{2} = \frac{1}{n!} \binom{n}{k} \frac{\pi}{2}
\]

Finally, summing (4) over \(k \geq 0\) yields Cesàro’s formula (1). The Bell numbers have many other pretty representations, including Dobinski’s infinite sum formula [3, p. 210]

\[
B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.
\]
References


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