On Generating Functions Involving the Square Root of a Polynomial

DAVID CALLAN
Department of Statistics
University of Wisconsin-Madison
1300 University Ave
Madison, WI 53706-1532
callan@stat.wisc.edu

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Abstract

Many familiar counting sequences, such as the Catalan, Motzkin, Schröder and Delannoy numbers, have a generating function (GF) that is algebraic of degree 2. For example, the GF for the central Delannoy numbers is $\frac{1}{\sqrt{1 - Ax + Bx^2}}$. Here we characterize GFs of the form $\frac{1}{\sqrt{1 + Ax + Bx^2}}$ that yield counting sequences and point out that they have a unified combinatorial interpretation in terms of colored lattice paths. We do likewise for the related forms $1 - \sqrt{1 + Ax + Bx^2}$ and $\frac{1 + Ax - \sqrt{1 + 2Ax + Bx^2}}{2Cx^2}$.

1 Introduction In this paper, all generating functions (GFs) are ordinary power series generating functions. Thus the GF for the formal power series $1 + x + x^2 + \ldots$ is $\frac{1}{1-x}$. A counting GF is one whose series expansion has nonnegative integer coefficients.

Many familiar counting GFs are algebraic of degree 2 and only involve the square root of a low-degree polynomial. A few such GFs are recalled in Table 1 below, with hyperlinks to the On-Line Encyclopedia of Integer Sequences [1].
Some Algebraic Generating Functions of Degree 2

<table>
<thead>
<tr>
<th>number sequence $(a_n)_{n \geq 0}$</th>
<th>first few terms</th>
<th>GF = $\sum_{n \geq 0} a_n x^n$</th>
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</thead>
<tbody>
<tr>
<td>even central binomial coefficients</td>
<td>1,2,6,20,70, ...</td>
<td>$\frac{1}{\sqrt{1-4x^2}}$</td>
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<tr>
<td>odd central binomial coefficients</td>
<td>1,3,10,35,126, ...</td>
<td>$\frac{1}{2x\sqrt{1-4x^2}} - \frac{1}{2x}$</td>
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<td>$1 - \frac{1}{2x}$</td>
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<tr>
<td>central trinomial coefficients</td>
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<td>$\frac{1}{\sqrt{1-4x^2}}$</td>
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<td>Motzkin numbers</td>
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<td>$1 - x - \frac{1}{2x} - \frac{3x^2}{2x}$</td>
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<tr>
<td>central Delannoy numbers</td>
<td>1,3,13,63,321, ...</td>
<td>$\frac{1}{\sqrt{1-6x+x^2}}$</td>
</tr>
<tr>
<td>big Schröder numbers</td>
<td>1,2,6,22,90, ...</td>
<td>$1 - x - \frac{1}{2x} - \frac{3x^2}{2x}$</td>
</tr>
<tr>
<td>little Schröder numbers</td>
<td>1,3,11,45,197, ...</td>
<td>$1 - 3x - \frac{1}{2x} - \frac{3x^2}{2x}$</td>
</tr>
</tbody>
</table>

Table 1

GFs of the form $\frac{1}{\sqrt{1+Ax+Bx^2}}$ and $\frac{1+x-Ax-\sqrt{1+2Ax+Bx^2}}{Cx^2}$ are prominent in Table 1. Our main results, Theorems 1 and 2 below, determine all counting GFs of these two forms and give a unified combinatorial interpretation for them in terms of colored lattice paths. We define and count the relevant lattice paths in §2, and complete the proofs of Theorems 1 and 2 in §3 using basic facts about orthogonal polynomials. §4 contains some concluding remarks.

The generic unital quadratic polynomial $1 + Ax + Bx^2$ can be written as $1 - 2ax + (a^2 - 4b)x^2$ with $a := -A/2$ and $b := (A^2 - 4B)/16$.

**Theorem 1.** Set $G_{a,b}(x) = \frac{1}{\sqrt{1-2ax+(a^2-4b)x^2}}$. Then

(i) $G_{a,b}(x) = \sum_{n \geq 0} \left( \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} a^{n-2k} b^k \right) x^n$,

(ii) $G_{a,b}(x)$ is a counting GF $\Leftrightarrow a, b$ are nonnegative integers,

(iii) when the conditions in (ii) hold, $G_{a,b}(x)$ is the GF for $a^F b^U$-colored trinomial paths with $x$ marking the number of steps.
Theorem 1 refers to exponent \(-1/2\) on the quadratic. The situation for exponent \(+1/2\) is a little more subtle. From the identity \(1 - \sqrt{1 - 2ax + (a^2 - 4b)x^2} = ax + 2b\sum_{n\geq0} \left( \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k a^{n-2k} b^k \right) x^n\) (proved below, \(C_k\) is the Catalan number), it is easy to see that this is a counting GF \(\Leftrightarrow a, b\) are nonnegative integers (sufficiency is obvious and necessity follows from just the first four coefficients: \(a, 2b, 2ab, 2a^2b + 2b^2\)). But then, also, it is clear that the greatest common divisor of all coefficients from the \(x^2\) term onward is \(2b\) and so it is natural to consider the refined GF \((1 - ax - \sqrt{1 - 2ax + (a^2 - 4b)x^2})/(2bx^2)\).

**Theorem 2.** Set \(F_{a,b}(x) = (1 - ax - \sqrt{1 - 2ax + (a^2 - 4b)x^2})/(2bx^2)\). Then

(i) \(F_{a,b}(x) = \sum_{n\geq0} \left( \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k a^{n-2k} b^k \right) x^n\),

(ii) \(F_{a,b}(x)\) is a counting GF \(\Leftrightarrow a, b\) are nonnegative integers,

(iii) when the conditions in (ii) hold, \(F_{a,b}(x)\) is the GF for nonnegative \(a^E b^U\)-colored trinomial paths with \(x\) marking the number of steps.

2 GFs for Colored Trinomial Paths A trinomial path is a lattice path of upsteps \(U = (1, 1)\), downsteps \(D = (1, -1)\) and flatsteps \(F = (1, 0)\) that ends at ground level, the horizontal line through its initial point. A trinomial \(n\)-path is one consisting of \(n\) steps. The name derives from the fact that the number of trinomial \(n\)-paths is clearly the constant term in \((x^{-1} + 1 + x)^n\), equivalently, the central trinomial coefficient \([x^n](1 + x + x^2)^n\). A nonnegative trinomial path, also known as a Motzkin path, is one that stays weakly above ground level. For \(a, b\) nonnegative integers, an \(a^E b^U\)-colored trinomial path is one in which each flatstep is colored with one of \(a\) specified colors and each upstep with one of \(b\) specified colors. Using the “symbolic” method [2] it is easy to obtain the GF, \(F(x)\), for \(a^E b^U\)-colored Motzkin paths with \(x\) marking the number of steps: the underlying path is either of the form \(F^i\ (i \geq 0)\) contributing \(a^i x^i\) to the GF, or \(F^i U PDQ\ (i \geq 0, P, Q\ arbitrary Motzkin paths)\) contributing \(a^i x^i b x^2 F^2\) to the GF. This yields

\[
F(x) = \sum_{i\geq0} a^i x^i + \sum_{i\geq0} a^i x^i b x^2 F(x)^2 = \frac{1 + b x^2 F(x)^2}{1 - ax},
\]

a quadratic equation for \(F(x)\) with (unique) solution

\[
F(x) = \frac{1 - ax - \sqrt{1 - 2ax + (a^2 - 4b)x^2}}{2bx^2}.
\]
The GF, \( G(x) \), for \( aFbU \)-colored trinomial paths is obtained similarly. The underlying path is (i) empty contributing 1 to the GF, or (ii) \( FP \) (\( P \) arbitrary trinomial path) contributing \( axG(x) \) to the GF, or (iii) \( UPDQ \) (\( P \) arbitrary Motzkin path, \( Q \) arbitrary trinomial path) contributing \( bx^2F(x)G(x) \) to the GF, or (iv) \( DPUQ \) (\( P \) arbitrary inverted Motzkin path, \( Q \) arbitrary trinomial path) also contributing \( bx^2F(x)G(x) \) to the GF. This leads to \( G(x) = 1 + axG(x) + 2bx^2F(x)G(x) \) whence

\[
G(x) = \frac{1}{\sqrt{1 - 2ax + (a^2 - 4b)x^2}}.
\]

On the other hand, it is also easy to count these colored paths directly by number of upsteps. For Motzkin \( n \)-paths containing \( k \) upsteps there are \( \binom{n}{2k} \) ways to position the slanted steps (\( U \) and \( D \)) among the \( n \) steps. There are \( C_k \) ways to arrange the slanted steps because they form a Dyck path [3, Ex. 6.19 (i), p. 221]. After applying colors, this yields a total of \( \binom{n}{2k}C_ka^{n-2k}b^k \) choices for \( aFbU \)-colored Motzkin \( n \)-paths containing \( k \) \( U \)s. The count for \( aFbU \)-colored trinomial \( n \)-paths is the same except that \( C_k \) must be replaced by \( \binom{2k}{k} \).

These results are enough to prove most of both Theorems 1 and 2: part (iii) and the “sufficiency” half of part (ii) obviously follow. Part (i) also follows because polynomials that agree on the nonnegative integers are identical. Alternatively, one could prove part (i) directly by setting \( a = 1 \) without loss of generality, equating coefficients of \( x^n \), and then using the automated WZ method [4] to verify the resulting identities. It remains only to prove “necessity” in part (ii).

### 3 An Application of Orthogonal Polynomials

The “necessity” half of part (ii) in Theorem 1 says: if \( p_n := \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} a^{n-2k}b^k \) is a non-negative integer for all \( n \geq 0 \), then \( a \) and \( b \) are nonnegative integers. Integrality of \( a \) and \( b \) follows immediately from the integrality of \( p_1 = a \) and \( p_2 = a^2 + b \). Similarly, the nonnegativity of \( a \) follows from that of \( p_1 \), but nonnegativity of \( b \) needs that of \( p_n \) for all \( n \) except in the trivial case \( a = 0 \). Assuming \( a > 0 \), we may without loss of generality set \( a = 1 \) and consider the polynomials \( p_n(b) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} b^k \). Now nonnegativity follows from

**Proposition 3.** \( p_n(b) \geq 0 \) for all \( n \) implies \( b \geq 0 \).

Clearly, \( p_2(b) = 1 + 6b < 0 \) on \(( -\infty, -1/6 ) \) and \( p_3(b) = 1 + 12b + 6b^2 < 0 \) on
\((-1.91\ldots, -0.08\ldots\), and we claim there exists a sequence of successively overlapping intervals \(I_n\) that cover \((-\infty, 0)\) such that \(p_n(b) < 0\) on \(I_n\). Proposition 3 follows. The claim in turn follows from the following facts about the zeros of \(p_n\).

**Proposition 4.** Let \(m\) denote \([n/2]\) so that \(\deg(p_n) = m\). Then

(i) the zeros of \(p_n\) are real and simple (no repeated roots), say \(b_{n1} < b_{n2} < \ldots < b_{nm} < 0\) (all zeros are obviously negative),

(ii) the zeros of \(p_n\) interlace those of \(p_{n+1}\), that is,

\[
b_{n1} < b_{n+1,1} < b_{n2} < b_{n+1,2} < \ldots < b_{n,n/2} < b_{n+1,n/2} \quad n \text{ even}
\]

\[
b_{n+1,1} < b_{n1} < b_{n+1,2} < b_{n2} < \ldots < b_{n+1,(n-1)/2} < b_{n,(n-1)/2} < b_{n+1,(n+1)/2} \quad n \text{ odd},
\]

(iii) for fixed integer \(k \geq 0\), \(b_{n,m-k} \to 0\) as \(n \to \infty\).

For the claim, use \(I_n = (b_{n,m-1}, b_{nm})\) and the case \(k = 0\) of part (iii).

Parts (i) and (ii) of Proposition 4 are reminiscent of orthogonal polynomials and indeed the \(p_n\) are closely related to the Legendre polynomials \(P_n(x)\) which are known to form an orthogonal polynomial sequence. Recall that the GF for the Legendre polynomials is

\[
\sum_{n \geq 0} P_n(x) w^n = \frac{1}{\sqrt{1 - 2xw + w^2}}
\]

while the GF for \(p_n\) is

\[
\sum_{n \geq 0} p_n(b) w^n = \frac{1}{\sqrt{1 - 2w + (1-4b)w^2}}.
\]

It follows that \(P_n\) and \(p_n\) are related by

\[
P_n(x) = x^n p_n \left(\frac{x^2 - 1}{4x^2}\right).
\]

The well known properties of orthogonal polynomials imply that the zeros of \(P_n(x)\) are real, simple and possess the interlacing property. Also, \(P_n\) is alternately even/odd and so its zeros are symmetric about 0. In particular, \(P_n\) has \(m := [n/2]\) positive zeros. If \((x_i)_{i=1}^m\) are the positive zeros of \(P_n\) in increasing order, then by (1), \(\left(\frac{1}{4}(1 - \frac{1}{x_i^2})\right)_{i=1}^m\) are the zeros of \(p_n\), also in increasing order. Parts (i) and (ii) of Proposition 4 follow.

Part (iii) is a simple consequence of (1), the fact that \(\cos \theta \to 0\) as \(\theta \to \pi/2\), and Bruns’ inequalities for the zeros of the Legendre polynomials, which show that the zeros
of \( P_n(\cos \theta) \) are fairly evenly spaced around the unit half-circle. (All zeros of \( P_n(x) \) lie in the interval \((-1, 1)\), as is evident from (1)).

**Bruns’ Inequalities** [5] Let \( \theta_1 < \theta_2 < \ldots < \theta_n \) denote the zeros of \( P_n(\cos \theta) \) in the interval \((0, \pi)\). Then

\[
\frac{j - \frac{1}{2}}{n + \frac{1}{2}} \pi < \theta_j < \frac{j}{n + \frac{1}{2}} \pi \quad j = 1, 2, \ldots, n.
\]

This completes the proof of Theorem 1.

Similarly, for Theorem 2 we must show that the following result holds for \( q_n(b) := \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} c_k b^k \).

**Proposition 5.** \( q_n(b) \geq 0 \) for all \( n \) implies \( b \geq 0 \).

To prove Prop. 5, we again obtain a sequence of overlapping intervals covering \((-\infty, 0)\) on the \( n \)th of which \( q_n \) is negative. Here we find

\[
Q_n(x) = (n + 1)x^n q_n \left(\frac{x^2 - 1}{4x^2}\right),
\]

where the \( Q_n(x) \) are the Jacobi polynomials \( J(n, 1, 1, x) \), which are also orthogonal. (The Legendre polynomial \( P_n(x) \) is \( J(n, 0, 0, x) \).) So, as before, the two largest zeros of \( q_n \) yield the overlapping intervals. Bruns’ inequalities fail here but since \( D_x(x q_n(x)) = p_n(x) \), the zeros of \( p_n \) separate those of \( q_n \), and Prop. 4 (iii) with \( k = 1 \) implies that the largest zero of \( q_n \to 0 \) as \( n \to \infty \) and so the overlapping intervals cover all of \((-\infty, 0)\).

This completes the proof of Theorem 2. Table 2 below contains some OEIS sequences whose GFs are of the types considered above.
<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>generalized Motzkin GF</th>
<th>generalized central trinomial GF</th>
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</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>[ \frac{1 - ax - \sqrt{1 - 2ax + (a^2 - 4b)x^2}}{2bx^2} ] generates this counting series</td>
<td>[ \frac{1}{\sqrt{1 - 2ax + (a^2 - 4b)x^2}} ] generates this counting series</td>
</tr>
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<td>1</td>
<td>Motzkin numbers</td>
<td>central trinomial coefficients</td>
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<tr>
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<td>central coeff ((1 + x + 3x^2)^n)</td>
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<td>central coeff ((1 + 2x + 3x^2)^n)</td>
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<td>lattice paths w/steps ((k, \pm k))</td>
<td>central coeff ((1 + 5x + 4x^2)^n)</td>
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<td>7</td>
<td>12</td>
<td>A098659</td>
<td>central coeff ((1 + 7x + 12x^2)^n)</td>
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</tbody>
</table>

Table 2

4 Further Remarks The middle coefficient of \((1 + ay + by^2)^n\) is the constant term in \((y^{-1} + a + by)^n\) and and it is easy to see directly that this is the number
of $a^F b^J$-colored trinomial $n$-paths as defined above. By Theorem 1, then, the GF for the middle coefficient of $(1 + ay + by^2)^n$ is $(1 - 2ax + (a^2 - 4b)x^2)^{-\frac{1}{2}}$. Graham, Knuth and Patashnik [6, p. 575] attribute this result to Herbert Wilf, citing his book *generatingfunctionology* [7] but it does not seem to be there!

There is a continued fraction expansion for $F_{a,b}(x)$ of Theorem 2:

$$
\frac{1 - ax - \sqrt{1 - 2ax + (a^2 - 4b)x^2}}{2bx^2} = \frac{1}{1 - ax - \frac{bx^2}{1 - ax - \frac{bx^2}{1 - ax - \ldots}}}
$$

This is easy to prove directly and is a special case of a general theorem on combinatorial interpretation of continued fractions [8].

**References**


